Kac–Dynkin diagrams and supertableaux

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We show the relation between Kac–Dynkin diagrams and supertableaux.

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I. INTRODUCTION

There exist presently two approaches to labeling representations of the supergroups $SU(N/M)$; Kac[1] has proposed a unique labeling of irreducible representations in analogy to Dynkin diagrams. We shall refer to these as Kac–Dynkin diagrams. Balentekin and Bars[2–4] introduced supertableaux in analogy to Young tableaux, which rely on the properties of the permutation group, to arrive at irreducible supertensors which provide a basis for supergroup representations. The relation between these approaches has been found.[5] Our aim is to elaborate further on this connection, add new insights and clarifications, and establish a useful dictionary.

A Kac–Dynkin diagram provides the highest weight Λ. The remaining weights are in principle obtained by applying lowering operators. This requires lengthy (but straightforward) calculations,[6] which yield the eigenvalues of the generators belonging to the Cartan subalgebra. With this method necessary and sufficient conditions as well as dimension formulas for "typical" representations have been given.[1,6] Also branching rules for supersubalgebras, especially irregular ones, have been computed.[6]

The supertableaux, and the associated supertensors, provide all the states (or modules) in a representation and the content of the states is immediately obvious. This makes them very useful in physical applications.[7,8] Typical and atypical representations are not distinguished in this approach and the supertableaux methods apply to both. In supertableaux one uses the concept of supersymmetrization,[2] which means that, when bosonic indices corresponding to a row are symmetrized, the fermionic indices are antisymmetrized. This can be done by an efficient method[2] which keeps close analogy to representations constructed via ordinary Young tableaux. These analogies can be applied as follows:

$$SU(N) \leftrightarrow SU(N/M),$$

$$SO(N) \leftrightarrow Osp(N/M),$$

$$Sp(2N) \leftrightarrow P(2N).$$

Through these analogies many practical and useful properties have been computed for the supergroups indicated above for all supertableau representations:

(i) matrix representations of the supergroup in tensor space;[2,4,7]

(ii) character formulas;[2,4]

(iii) dimension formulas;[2,4]

(iv) eigenvalues of Casimir operators;[2–4]

(v) branching rules for[2,4]

$$SU(M/N) \rightarrow SU(M) \times SU(N) \times U(1),$$

$$SU(M_1 + M_2/N_1 + N_2) \rightarrow SU(M_1/N_1) \times SU(M_2/N_2) \times U(1),$$

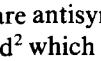
$$SU(M_1M_2 + N_1N_2/M_1N_2 + M_2N_1) \rightarrow SU(M_1/N_1) \times SU(M_2/N_2);$$

(vi) harmonic oscillator representations.[4,7,9,10]

(vii) analytic unitary reprsentations of noncompact $SU(M, P/N + Q)$ in a harmonic oscillator basis[10] and in a superspace Z-basis.[10]

The connection to Kac–Dynkin diagrams[5] for $SU(M/N)$ can be seen by computing the highest weight through the aid of the $SU(M/N) \rightarrow SU(M) \times SU(N) \times U(1)$ decomposition. In this paper after reviewing this procedure and giving a translation dictionary to Kac–Dynkin diagrams, and several examples, we will be able to establish the following statements for $SU(M/N)$:

(a) Supertableaux containing only covariant (undotted) or only contravariant (dotted) boxes correspond to irreducible representations.

(b) Supertableaux containing mixed dotted and undotted boxes correspond to irreducible representations provided M, N are sufficiently large compared to the number of boxes. The irreducibility of some supertableaux requires conditions also on N-M. For example,  is irreducible for all $N \neq M$, but reducible for $N = M$, as noted in Refs. 2 and 4.

(c) Mixed supertableaux with too many boxes compared to M,N are generally reducible but *indecomposable*!

(d) All atypical representations are described by supertableaux.

(e) Typical representations with a_M = integer (defined below) are naturally described. a_M = arbitrary real number is described with the additional concept of an overall U(1) phase of the representation in addition to the tableau.

(f) To a given Kac–Dynkin diagram one can find many corresponding supertableaux.

(g) One can usefully employ supertableaux to compute the decomposition of direct product of any representations, provided indecomposable supertableaux are reduced via Kac–Dynkin diagrams.

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2. THE SUPERALGEBRA SU(M/N)

In the classification of Kac,[1,11] this is a classical superalgebra of type I, called A (M-1, N-1). SU(M/N) is simple for $M \neq N$. For $M = N$, one has to divide by U(1). It consists of an even ("Bosonic") part, the subalgebra SU(M)×SU(N)×U(1) and an odd ("fermionic") part, which transforms as the representation $(M, N^*) + (M^*, N)$ of the even part. The Cartan subalgebra consists of the $M + N - 1$ mutually commuting generators H_i, the $M - 1$ first ones belonging to SU(M), the $N - 1$ last ones to SU(N), H_M playing a special role. The generator Q of U(1) is a linear combination of H_i [see Eq. (2.10) below]. To each H_i corresponds a simple root a_i, a "raising" operator E_i^+ and a "lowering" operator E_i^-. We shall need the commutation relations

$$[H_i, E_j^{\pm}] = \pm a_{ij} E_j^{\pm}, \quad i,j = \cdots M+N-1, \tag{2.1}$$

where a_{ij} are the elements of the Cartan matrix of SU(M/N) given by Kac[1]

$$\{a_{ij}\} = \begin{array}{c|ccccc|c|ccccc|} & & & & & & M & & & & & \\ & 2 & -1 & & & & & & & & & \\ & -1 & 2 & -1 & & & & & & & & \\ & & -1 & \ddots & & & & & & & & \\ & & & & -1 & 2 & -1 & & & & & \\ \hline & & & & & -1 & 0 & +1 & & & & M \\ \hline & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & -1 & & \\ & & & & & & & & & -1 & 2 & \end{array} \tag{2.2}$$

Notice that $a_{MM} = 0$ and $a_{M\,M+1} = +1$, otherwise we recognize the Cartan matrices of SU(M) and SU(N).

We also note the commutation relations

$$[E_i^+, E_j^-] = \delta_{ij} H_j \quad \forall i,j \neq M \tag{2.3}$$

and the anticommutation relations of the two odd generators corresponding to the simple root a_M:

$$\{E_M^+, E_M^-\} = H_M. \tag{2.4}$$

The full system of commutation (anticommutation) relations can either be obtained from (2.1)–(2.4), which characterize "simple" generators, plus the generalized Jacobi identity,[1,11] or by the explicit realization of the fundamental representation of dimension $M + N$. This will now be done. The generators are the matrices X:

$$X = \begin{array}{c|c|} \overbrace{}^{M} & \overbrace{}^{N} \\ \hline A & B \\ \hline C & D \\ \hline \end{array} \begin{array}{l} \} M \\ \} N \end{array} \tag{2.5}$$

with the restriction for the supertrace

$$\mathrm{Str}\, X \equiv \mathrm{tr}\, A - \mathrm{tr}\, D = 0. \tag{2.6}$$

Introducing the matrices E^i_j with matrix elements

$$(E^i_j)^a_b = \delta^{ia}\delta_{jb} \quad i,j,a,b = 1 \cdots M+N, \tag{2.7}$$

one gets for the Cartan subalgebra

$$H_i = E^i_i - E^{i+1}_{i+1}, \quad i = 1 \cdots M-1, M+1 \cdots M+N-1, \qquad H_M = E^M_M + E^{M+1}_{M+1}; \tag{2.8}$$

all H_i have zero supertrace.

The raising operators are E^i_j, $i<j$, the lowering operators E^i_j, $i>j$, and the "simple" generators of Eq. (2.1) correspond to $j = i + 1$, resp. $i - 1$. For the odd generators, $i \leqslant M$, $j > M$ or $i > M$, $j \leqslant M$. Notice that the anticommutator of two odd raising or two odd lowering operators is zero.

Finally, the generator Q of U(1) is given, up to a multiplicative constant, by

$$Q = \begin{array}{|ccc|} \hline 1/M & & \\ & \ddots & \\ & 1/M & \\ \hline & & 1/N \\ & & \ddots \\ & & 1/N \\ \hline \end{array}, \tag{2.9}$$

which corresponds to

$$Q = \sum_{k=1}^{M-1} \frac{kH_k}{M} + H_M - \sum_{l=1}^{N-1} \frac{(N-l)}{N} H_{M+l}. \tag{2.10}$$

From (2.1) and (2.2) one gets

$$[Q, E_M^{\pm}] = \pm\left(\frac{M-1}{M} a_{M-1,M} - \frac{N-1}{N} a_{M+1,M}\right) E_M^{\pm} = \pm\left(\frac{1}{M} - \frac{1}{N}\right) E_M^{\pm}. \tag{2.11}$$

3. THE KAC–DYNKIN DIAGRAM

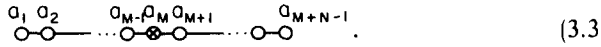
According to Kac,[1] the irreducible representations (IR) of the superalgebra SU(M/N) are characterized in a similar way as IR of Lie algebras. They are uniquely determined by the highest weight Λ, which is a vector in the root space. The state in the representation space corresponding to Λ is defined by

$$E_i^+ |\Lambda\rangle = 0, \quad i = 1 \cdots M+N-1, \tag{3.1}$$

$$H_i |\Lambda\rangle = a_i |\Lambda\rangle, \quad i = 1 \cdots M+N-1. \tag{3.2}$$

The numbers a_i are nonnegative integers for $i \neq M$. a_M may be any real number.

An IR of SU(M/N) is thus defined by the values a_i of the highest weight, which can be noted on a Kac–Dynkin diagram



(3.3)

The part without ⊗ decomposes into ordinary Dynkin diagrams for SU(M) and SU(N). ⊗ corresponds to the odd root a_M (whose length is zero!), or to the special generator H_M.

One distinguishes typical and atypical IR. For the latter, *one* of the following conditions must be satisfied (Kac, Ref. 1, Hurni and Morel, Ref. 6):

$$a_M = \sum_{i=M+1}^{j} a_i - \sum_{i=1}^{M-1} a_i - 2M + i + j, \qquad (3.4)$$

$$1 \leqslant i \leqslant M \leqslant j \leqslant M + N - 1.$$

For the typical representations, *none* of these relations is satisfied. Their interpretation is the following. One gets all the weights of a given IR by starting with the highest weight and applying lowering operators. The action of the even operators is well known. There are MN odd generators which anticommute. Hence, each one can be applied at most once, and the state obtained by applying two different odd generators is antisymmetric. If $|\psi\rangle$ is some state in the representation space, it may happen that $E_M^+ E_M^- |\psi\rangle = 0$. This is just the case when one of the relations (3.4) is satisfied. For example, if $a_M = 0$, $E_M^+ E_M^- |\Lambda\rangle = 0$, where Λ is the highest weight. This means that the state $E_M^- |\psi\rangle$ does not belong to the same representation: Either the representation starting with $|\Lambda\rangle$ is not irreducible, or we must put $E_M^- |\psi\rangle = 0$. This is the atypical case.

For typical representations, one can apply each odd generator exactly once. If d is the dimension of the IR of $SU(M) \times SU(N) \times U(1)$ corresponding to the highest weight Λ, the dimension D of the corresponding IR of $SU(M/N)$ is

$$D = 2^{MN} d, \qquad (3.5)$$

with an equal number of bosons and fermions.
For atypical representations, the dimension will always be lower.

The fundamental representation ($D = M + N$) is given by

(3.6)

Since $a_M = 0$, it is atypical (put $i = j = M$ into Eq. (3.4)). For $SU(1/N)$, we have:

(3.7)

Here $a_M = a_1 = a_2 + 1$, which satisfies again (3.4) (put $i = M = 1, j = M + 1 = 2$).

The conjugate representation ($D = M + N$) is

(3.8)

and is again atypical, also for $SU(M/1)$.

The adjoint representation is

(3.9)

This is atypical, except for SU(1/2):

(3.10)

whose dimension is $2 \cdot 2^2 = 8$.

Another convenient characterization of the highest weight Λ is obtained by considering the eigenvalue q of the U(1) generator Q:

$$Q|\Lambda\rangle = q_\Lambda |\Lambda\rangle. \qquad (3.11)$$

Using (2.10) and (3.2), one gets

$$q_\Lambda = \sum_{k=1}^{M-1} \frac{k a_k}{M} + a_M - \sum_{l=1}^{N-1} \frac{(N-l) a_{M+l}}{N}. \qquad (3.12)$$

Applying odd lowering operators, one gets the other $SU(M) \times SU(N) \times U(1)$ multiplets. Each odd operator is obtained by the commutator of E_M^- with "simple" even generators. Since the latter commute with Q, it is enough to consider the commutator given by Eq. (2.11), applied on a state $|\psi\rangle$:

$$[Q_1 E_M^-]|\psi\rangle = Q E_M^- |\psi\rangle - q_\psi E_M^- |\psi\rangle$$
$$= -(1/M - 1/N) E_M^- |\psi\rangle. \qquad (3.13)$$

Thus, for $M < N$, resp. $M > N$, the odd generators E_M^- lowers, resp. raises the value of q_ψ. Hence, the "highest" weight Λ corresponds to

$$q_\Lambda = \text{maximum} \quad \text{for} \quad M < N,$$
$$q_\Lambda = \text{minimum} \quad \text{for} \quad M > N. \qquad (3.14)$$

For typical representations, one has to apply the MN odd generators in a completely antisymmetric way to get the lowest weight λ. Since such an antisymmetric combination is a singlet under $SU(M) \times SU(N)$, the state $|\lambda\rangle$ belongs to the same IR of this subalgebra. The eigenvalue q_λ is given, using (3.13), by

$$q_\lambda = q_\Lambda + M - N, \qquad (3.15)$$

where

$$Q|\lambda\rangle = q_\lambda |\lambda\rangle, \quad E_i^- |\lambda\rangle = 0 \quad \forall\ i. \qquad (3.16)$$

For atypical representations, q_λ will be different from the expression (3.15), namely, larger if $M < N$ and smaller if $M > N$.

In conclusion, the Kac–Dynkin diagram characterizes uniquely all IR of $SU(M/N)$. It gives immediately the eigenvalues of H_i and Q of the highest weight. It allows a usually lengthy but straightforward computation of all states of the representation. It gives immediately the dimension and $SU(M) \times SU(N) \times U(1)$ content of typical representations, but not those of atypical representations.

4. YOUNG SUPERTABLEAUX FOR SU(M/N)

Young tableaux for Lie algebras are very convenient for computing branching rules for representations of subalgebras and for establishing the Clebsch–Gordan series of tensor products of IR. They are very useful in practical physics applications because it is possible to describe states in tensor notation with the symmetries of Young tableaux.

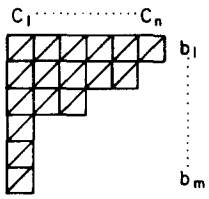
Balantekin and Bars (BB)[2–4] have introduced Young supertableaux for $SU(M/N)$ and showed that these, in addition to providing a very convenient labeling of representations, are useful in calculating many properties of super-representations.

For $SU(M)$, Young tableaux give the symmetry of the indices of covariant tensors $t_{ABC\ldots}$. One can also introduce contravariant tensors $t^{A'B'C'\cdots}$. They are related to the former ones by the completely antisymmetric symbol $\epsilon_{A_1,\ldots,A_M}$, which is invariant due to the determinant of $SU(M)$ group elements being one. Although this is not necessary, King[12]

has introduced Young tableaux for contravariant tensors (distinguished graphically by a point in the box). A next step is to introduce traceless mixed tensors $t_{AB\dots}^{A'B'\cdots}$.

For SU(M/N), the ϵ symbol is not invariant. Thus both co- and contravariant tensors are necessary. These correspond to mixed supertableaux. Furthermore, it is possible to have tensors corresponding to long columns in the supertableaux with more than $M+N$ dotted or undotted boxes.

BB[2,3] assign to the covariant tensor $t_{AB\dots}$ the Young supertableaux:

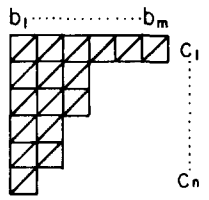


(4.1)

where the $b_i (i = 1,\dots,m)$ counts the boxes in the row i and c_j $(j = 1,\dots n)$ counts the boxes in the column j, with the conditions

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m > 0,$$
$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n > 0. \tag{4.2}$$

The conjugate tableau is obtained by interchanging rows and columns:

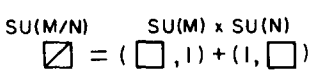


(4.3)

The supersymmetry property of $t_{AB\dots}$ under interchange of the indices $A, B, \cdots$ is analogous to SU($M+N$) except that when bosonic indices in a row are symmetrized, fermionic indices are antisymmetrized. This is the meaning of supersymmetrization.

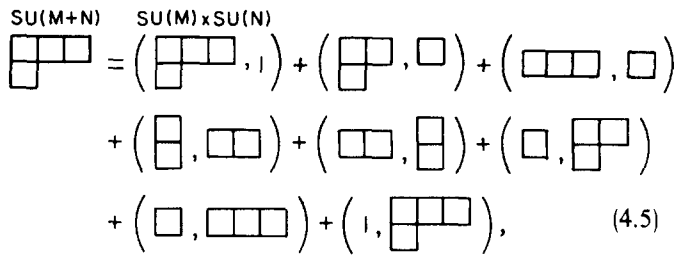
Consider now the IR of the subalgebra SU(M) $\times$SU(N)$\times$U(1) contained in an IR of SU(M/N). The procedure to get these IR is the same as for SU($M+N$), with the essential difference that the tableau one would obtain for an IR of the second algebra SU(N) has to be replaced by the conjugate tableau. This follows from supersymmetrization. For details, see the third paper in Ref. 2.

Starting from fundamental representation (dimension $M+N$)

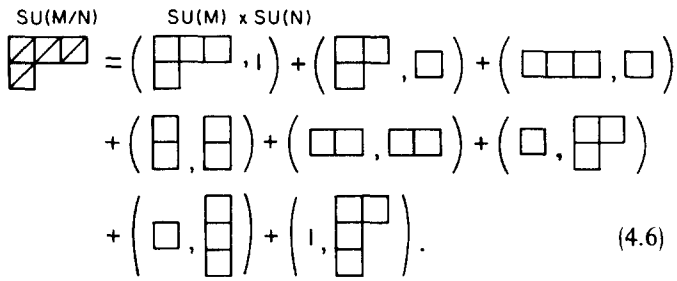


(4.4)

the rule is shown in the following example where the decomposition of an IR of SU($M+N$) is compared to the decomposition of an IR of SU(M/N):



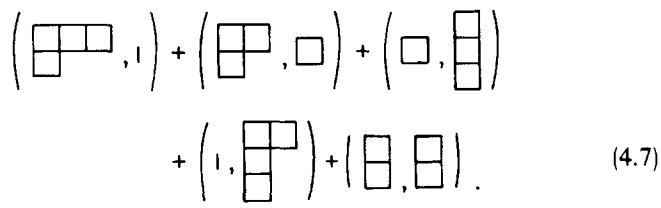
(4.5)



(4.6)

This rule is easy to understand in tensor notation.[2] Also from the point of view of the algebra, each time one replaces an SU(M) index by an SU(N) index, one has to apply an odd generator. Since the product of odd generators is antisymmetric, rows (symmetric) are changed in to columns (antisymmetric) and vice versa.

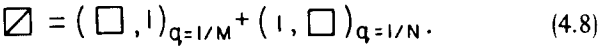
Equations (4.4) and (4.6) are independent of M and N, except if M or (and) N are too small. For example, for SU(1/2), the following terms are illegal, applying the rules for SU(M)$\times$SU(N):



(4.7)

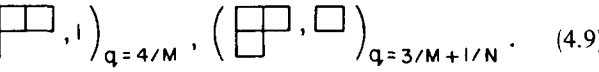
They must be dropped since M and/or N are too small.

The eigenvalue q of the U(1) generator Q is obtained from Eq. (2.9). Thus for the fundamental representation



(4.8)

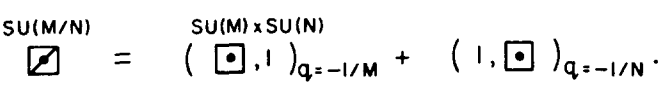
Hence the q value of some SU(M)$\times$SU(N) IR is given by $1/M$ times the number of SU(M) boxes plus $1/N$ times the number of SU(N) boxes. For example, for the first two terms of (4.6), one gets



(4.9)

Notice that the difference of these two q values agrees with (3.13).

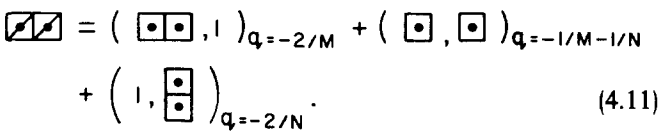
Contravariant tensors $t^{AB\cdots}$ correspond to conjugate representations of SU(M/N), as well as for the subalgebra SU(M)$\times$SU(N)$\times$U(1). The fundamental conjugate IR is denoted by



(4.10)

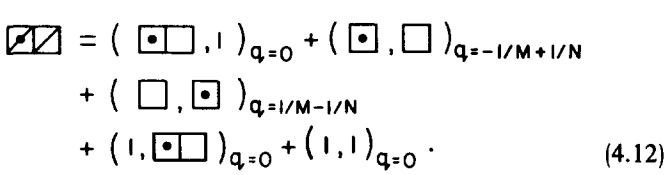
Both q values are negative because the supertrace of Q must be zero [Eq. (2.6)].

Apart from this, the rules are similar as for the covariant tensor. For example,



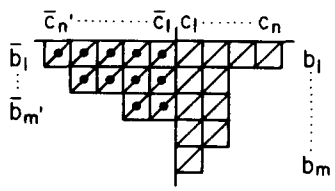
(4.11)

Finally, mixed tensors $t^{A'B'\cdots}_{AB\cdots}$ correspond to IR only if the supertrace is zero, $\Sigma_x(-1)^{g(x)}t^{XB'\cdots}_{XB\cdots}=0$, where $g(x)=0$ for even, and $g(x)=1$ for odd components. This is the case for the adjoint representation



(4.12)

The notation for the general supertableau for SU(M/N) will be[3]



(4.13)

5. RELATIONS BETWEEN KAC–DYNKIN DIAGRAMS AND YOUNG SUPERTABLEAUX

Kac[1] has used the highest weight to uniquely determine an IR of SU(M/N). What is the relation between Kac–Dynkin diagrams and supertableaux?

From (3.14) we know the properties of the eigenvalue q_Λ of Q for the state corresponding to the highest weight Λ:

$$q_\Lambda = \text{maximum} \quad \text{for } M<N, \tag{5.1}$$

$$q_\Lambda = \text{minimum} \quad \text{for } M>N.$$

The case $M=N$ will not be considered.

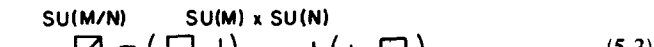
From (3.12) we know the relation to the Dynkin labels a_i:

$$q_\Lambda = \sum_{k=1}^{M-1}\frac{ka_k}{M} + a_M - \sum_{l=1}^{N-1}\frac{N-l}{N}a_{m+l}. \tag{5.2}$$

We now need only the corresponding information for supertableaux.

Let us start with tableaux corresponding to covariant tensors [see tableau (4.1)].

From (4.8) we know for the fundamental representation:

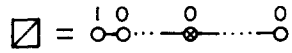


(5.3)

From (5.1) we see that the highest weight belongs to (□, 1). From (5.2) it is clear that

$$a_1=1; \quad a_i=0, \quad i\neq 1. \tag{5.4}$$

Hence



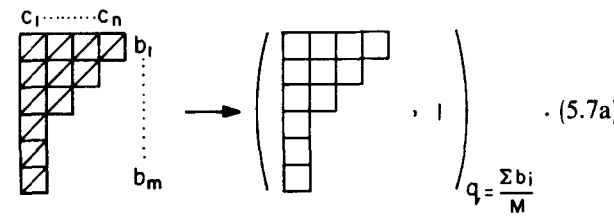
(5.5)

in agreement with (3.6).

For a general covariant tableau, one gets q by counting boxes. From (5.1) and (5.3) one sees that q_Λ is obtained with the maximum number of SU(M) boxes. If c_1, the number of rows, does not exceed M, the number of SU(M) boxes can be taken to be equal to the number of SU(M/N) boxes. The Dynkin labels are given by the familiar formula for SU(M), while a_M is fixed by (5.2), remembering that, for SU(M), a_K is the number of columns with K boxes:

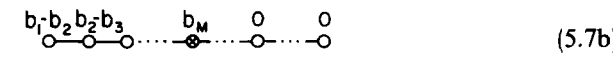
$$\begin{aligned} &a_i = b_i - b_{i+1}, \quad i=1\cdots M-1,\\ &a_M = b_M,\\ &a_{M+j}=0, \quad j=1\cdots N-1\\ &c_1\leqslant M. \end{aligned} \tag{5.6}$$

In pictures,



(5.7a)

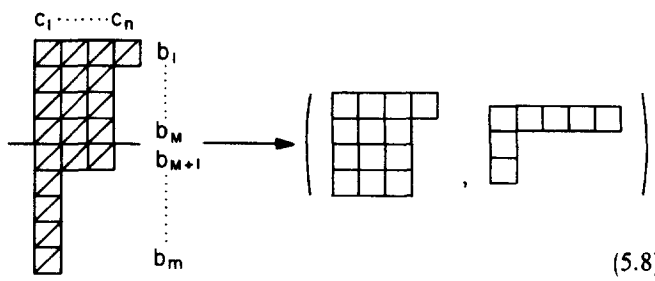
where in the SU(M)×SU(N)×U(1) decomposition we have shown just the component with the value q corresponding to the highest weight. This immediately yields



(5.7b)

if $c_1=m\leqslant M$. If c_1 exceeds M, to determine the highest state, one writes first the step of Eq. (5.7a), and then one has to cut the supertableau in two pieces. The first piece, which contains the first M rows, is assigned to SU(M), and the remaining rows are assigned to SU(N), after conjugating them. The SU(M)×SU(N)×U(1) Young tableau thus defined is the first nonvanishing component in the decomposition of SU(M/N) → SU(M)×SU(N)×U(1), which will have the right value q corresponding to the highest state.

For example



(5.8)

On the rhs we have shown the IR of SU(M)×SU(N) corresponding to the highest weight. The value of q_A is

$$q_A = \sum_{i=1}^{M} \frac{b_i}{M} + \sum_{i=1}^{b_{m+1}} \frac{c_i - M}{N} \theta(c_i - M), \tag{5.9}$$

where only $c_i > M$ contributes.

As far as the SU(M) content is concerned, one can subtract SU(M) singlets for each column with M boxes. On the other hand, a supertableau is illegal unless

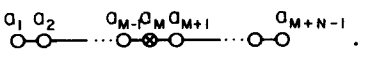
$$b_{M+1} \leqslant N, \tag{5.10}$$

because otherwise every component in the decomposition vanishes.

With (5.2), (5.9), and the rules for SU(M) and SU(N), we get the generalization of (5.6):

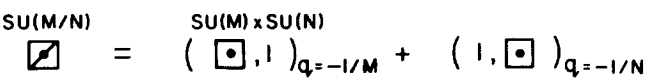
$$\begin{aligned} &a_i = b_i - b_{i+1}, \quad i = 1 \cdots M-1, \\ &a_M = b_M + c'_1, \\ &a_{M+j} = c'_j - c'_{j+1}, \quad j = 1 \cdots N-1, \\ &c'_j = (c_j - M)\theta(c_j - M), \\ &b_{M+1} \leqslant N. \end{aligned} \tag{5.11}$$

These values must be put on the Dynkin diagram



One should not forget that the b_i and c_j are not independent.

For conjugate representations, the procedure is similar except for the sign changes. For the fundamental representation, Eq. (4.1) is

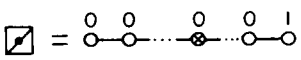


(5.12)

From (5.1), we see that the highest weight has

$$q_A = -1/N. \tag{5.13}$$

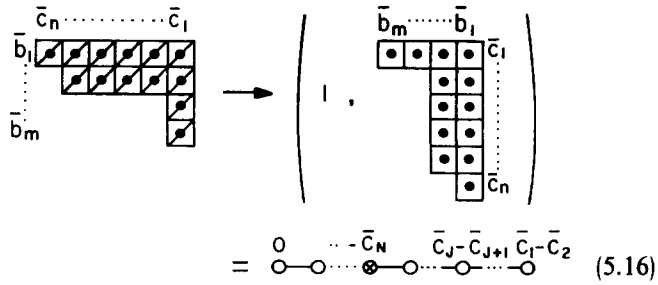
Hence, comparing (3.8) and (5.12),



(5.14)

For a general tableau, we search for q_A using (5.1) and (5.13). If $\bar{b}_1$, the number of columns, is smaller or equal to N, we can fulfill (5.1) with SU(N) boxes only. Using again (5.2) and the rules for SU(N), not forgetting to conjugate the SU(N) tableau, we get

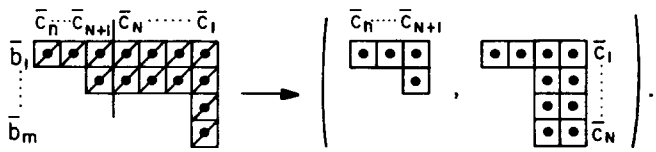
$$\begin{aligned} &a_i = 0, \quad i = 1 \cdots M-1, \\ &a_M = -\bar{c}_N, \\ &a_{M+N-j} = \bar{c}_j - \bar{c}_{j+1}, \quad j = 1 \cdots N-1, \\ &n = \bar{b}_1 \leqslant N, \end{aligned} \tag{5.15}$$



(5.16)

if $n = \bar{b}_1 \leqslant N$.

Next consider the case where $\bar{b}_1$ exceeds N. We now must cut the supertableau in two pieces by a vertical line. If j is the index of $\bar{c}_j$, the contribution for $j < N$ is as before. For $j > N$, one gets an SU(M) IR. Thus,



(5.17)

The tableau on the rhs again corresponds to the highest weight of the SU(M/N) IR. The value of q_A is

$$q_A = -\sum_{i=1}^{N} \frac{\bar{c}_i}{N} - \sum_{i=1}^{\bar{c}_{N+1}} \frac{(\bar{b}_i - N)}{M} \theta(\bar{b}_i - N). \tag{5.18}$$

A supertableau is illegal unless

$$\bar{c}_{N+1} \leqslant M. \tag{5.19}$$

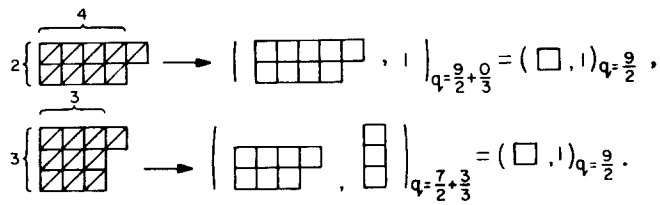
Using (5.2) and (5.18), we get for the Dynkin labels a_i,

$$\begin{aligned} &a_{M-i} = \bar{b}'_i - \bar{b}'_{i+1}, \quad i = 1 \cdots M-1, \\ &\bar{b}'_i = (\bar{b}_i - N)\theta(\bar{b}_i - N), \\ &a_M = -\bar{c}_N - \bar{b}'_1 = -\bar{c}_N - (\bar{b}_1 - N)\theta(\bar{b}_1 - N), \\ &a_{M+N-j} = \bar{c}_j - \bar{c}_{j+1}, \quad j = 1 \cdots N-1 \\ &\bar{c}_{N+1} \leqslant M \end{aligned} \tag{5.20}$$

6. DISCUSSION

We have shown that to each covariant tensor (with a corresponding legal supertableau) one can sssign a Kac–Dynkin diagram. The latter, we know, specifies an IR of SU(M/N), for $M \neq N$. The same is true for contravariant tensors. The case of mixed tensors will be considered in the next section.

We now show that this correspondence is not one-to-one. Take, for example, SU(2/3). Consider the following two supertableaux and their highest weight



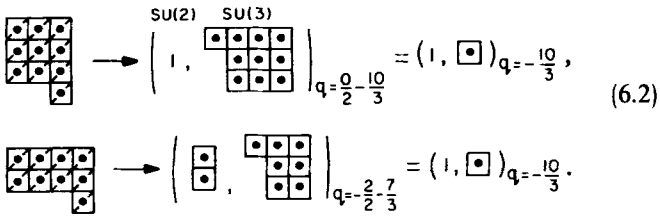
(6.1)

They clearly correspond to the same Kac–Dynkin diagrams.

For SU(M/N), we find the rule: Two supertableaux correspond to the same Kac–Dynkin diagram if $(N+1)$ columns of M boxes are replaced by N columns of $M+1$ boxes. This amounts to replacing an SU(M) singlet with $q = M/M$ by an SU(N) singlet with $q = N/N$, provided there are enough boxes to start with.

This ambiguity is, of course due to the fact that (5.11) does not determine the b_i's and c_j's uniquely from a_i.

A similar rule applies to contravariant tensor (5.20). Consider for SU(2/3) the supertableaux

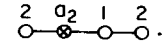


(6.2)

N columns of $M+1$ boxes are replaced by $N+1$ columns of M boxes.

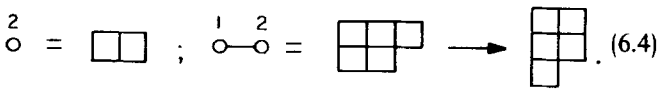
Consider now the inverse problem: Given Dynkin labels a_i, calculate supertableaux labels b_i. There arises a question: For typical representations, a_M can be any real number while the supertableau describes naturally a_M = integer since the value of Q is determined by an integer number of boxes. However, for typical representations it is possible to add any constant to Q, since it remains supertraceless when the number of bosons is equal to the number of fermions. An additional constant in Q corresponds to an overall U(1) phase of the whole representation. This U(1) commutes with SU(M/N). Thus, up to this overall phase an arbitrary representation of the group is recovered through the supertableau. The role of this overall phase and its significance in representation theory of supergroups is not sufficiently clear.

Keeping this in mind, we start from a Kac–Dynkin diagram, and consider first the SU(M)×SU(N) labels a_i ($i = 1,\ldots,M-1, M+1,\ldots,M+N-1$) which specify the highest weight Λ of an IR. To each set a_i, we can assign either a covariant or a contravariant tensor. The general formulae are, of course, (5.11) and (5.20). To show how they work, it is best to give an example. Consider the algebra SU(2/3), and the diagram



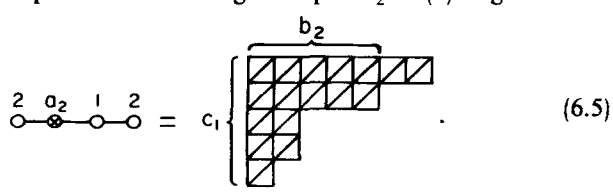
(6.3)

For the subalgebra SU(2)×SU(3), this corresponds to covariant tensors with tableaux:



(6.4)

For the SU(3) part, we also indicate the conjugate tableau. The supertableau is now given up to b_2 SU(2) singlets:



(6.5)

Here, $c_1 = 5$ and b_2 determined by a_2, using (5.11):

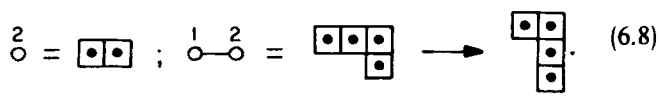
$$a_2 = b_2 + 3. \tag{6.6}$$

To get a legal diagram, $b_2 \geqslant 2$. Hence, (6.5) can be fulfilled for a_2 satisfying

$$a_2 \geqslant 5, \tag{6.7}$$

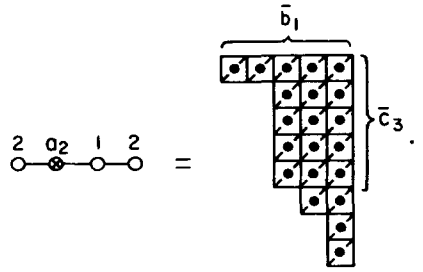
modulo the additional constant mentioned above, if the representation is typical.

Consider now contravariant tensors with tableaux [compare with (5.20)]



(6.8)

The supertableau is given up to $\bar{c}_3$ SU(3) singlets:



(6.9)

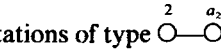
With (5.20), we get

$$a_2 = -\bar{c}_3 - 2. \tag{6.10}$$

Here, $\bar{c}_3 \geqslant 1$, so that a_2 satisfies

$$a_2 \leqslant -3 \tag{6.11}$$

up to the constant mentioned above.

We will see that we can also use mixed supertableaux to obtain representations of type  for SU(2/3).

Typical representations are those for which a_M is different from the rhs of (3.4):

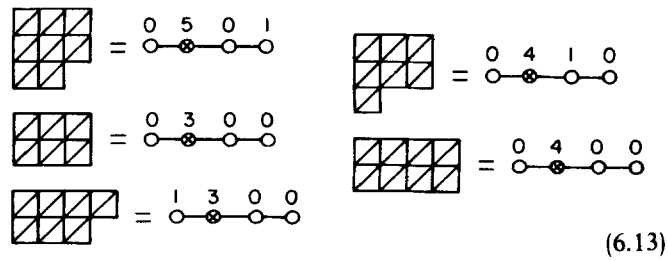
$$a_M \neq \sum_{i=M+1}^{j} a_i - \sum_{i=1}^{M-1} a_i - 2M + i + j, \tag{6.12}$$

$$1 \leqslant i \leqslant M \leqslant j \leqslant M+N-1.$$

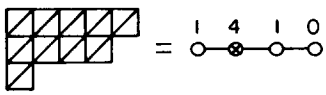
A necessary and sufficient condition for covariant tensors is

$$b_M \geqslant N.$$

This follows from the more general discussion in the next section. For example, for SU(2/3), the following are typical IR:



(6.13)

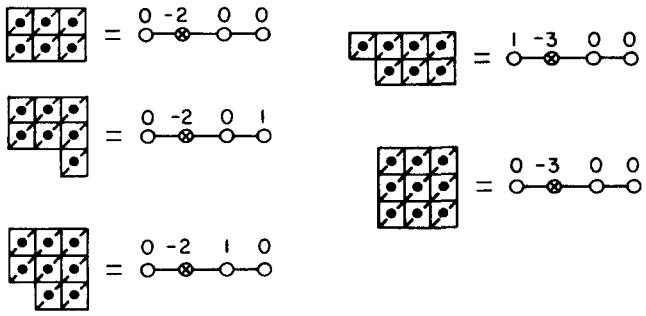


etc.

Similarly, for contravariant tensors,

$\bar{c}_N \geqslant M.$

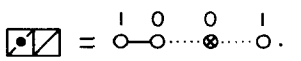
For example, for SU(2/3)



There are, of course, many more atypical representations than typical, and the supertableau approach is a convenient tool to describe both.

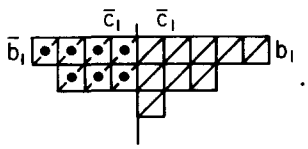
7. MIXED REPRESENTATIONS

We have seen that for covariant or contravariant tensors, a_M is limited by inequalities of the type (6.7) or (6.11). To get more general situations, one needs mixed, traceless tensors $t_{AB\ldots}^{A'B'\cdots}$. The first important example is the adjoint representation [see (3.9) and (4.12)].



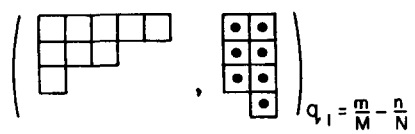
(7.1)

The algebraic rules to go from supertableau labels b_i to Dynkin labels a_i. are obtained again from Eqs. (5.11) and (5.20). There is, however, one essential complication: Mixed supertableaux, while being irreducible when N, M are sufficiently large compared to the number of boxes, may not always correspond to irreducible representations of SU(M/N), when N, M are small. But we shall see that they are indecomposable even when they are reducible. [One of us (I.B.) thanks V. Kac for his comment on this point.] Consider the general supertableau



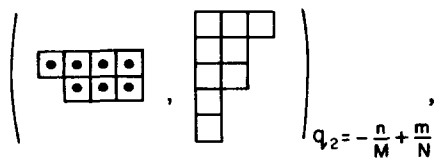
(7.2)

Suppose it contains m "covariant" boxes ☐ and n contravariant boxes ⊡. Consider now the two SU(M)×SU(N)×U(1) tableaux, obtained from (7.2):



(7.3)

and



(7.4)

$$q_2 = q_1 - (m+n)(1/M - 1/N). \tag{7.5}$$

Transferring a box ☐ from left to right, or a box ⊡ from right to left, amounts to applying an odd lowering operator. From (3.13)

$$QE_{\bar{M}}|\psi\rangle = [q_\psi - (1/M - 1/N)]E_{\bar{M}}|\psi\rangle \tag{7.6}$$

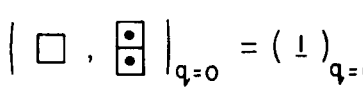
we see that (7.4) is obtained from (7.3) by applying $(m+n)$ different odd lowering generators. Now, if all tableaux in (7.3) and (7.4) are legal, these correspond to the state $|\Lambda\rangle$, resp. $|\lambda\rangle$ with highest, resp. lowest, weight. But for an IR, one can apply at most MN odd lowering generators to $|\Lambda\rangle$.

Hence, if $m+n > MN$, i.e., if there are too many boxes, (7.2) does not correspond to an IR of SU(M/N).

The simplest example is given by the supertableau of SU(1/2):

$$m+n=3, \quad MN=2. \tag{7.7}$$

The highest weight belongs to the IR of the bosonic subalgebra U(1)×SU(2):



(7.8)

The corresponding Kac–Dynkin diagram would be

(7.9)

But (7.7) gives rise to series of U(1)×SU(2) IR:

$$(\underline{1})_{q=0} + (\underline{2})_{q=-1/2} + (\underline{2})_{q=-1/2} + (\underline{3})_{q=-1} + (\underline{1})_{q=-1} + (\underline{1})_{q=-1} + (\underline{2})_{q=-3/2}. \tag{7.10}$$

We see that we need indeed three odd generators to go from $q=0$ to $q=\frac{3}{2}$, which is impossible for an IR.

Another way to see the reducibility is to try to construct the supertraceless tensor corresponding to the supertableau (7.7), as in Ref. 2, for SU(M/N)

$$t_C^{(AB)} = \varphi_C \varphi^{(AB)} - \frac{1}{M-N+1}[\delta_C^A(-)^{g(D)}\varphi_D\varphi^{(DB)} + (-)^{g(A)g(B)}\delta_C^B(-)^{g(D)}\varphi_D\varphi^{(DA)}] \tag{7.11}$$

such that the supertrace is zero:

$$\sum_{C=1}^{M+N}(-)^{g(C)}t_C^{(CB)} = 0,$$

$$g(C)=0, \quad C=1\cdots M,$$

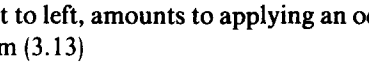
$$g(C)=1, \quad C=M+1\cdots M+N. \tag{7.12}$$

However, when $N=M+1$, e.g., for SU(1/2), the denominator vanishes, so that

$$t_C^{(AB)} = t_C^{(BA)}(-1)^{g(A)g(B)} \tag{7.13}$$

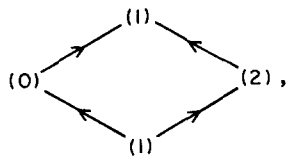
contains an invariant subspace which cannot be subtracted. This means that the tensor is reducible but *indecomposable*!!

Studying the weight diagram of (7.10) in more detail, one finds the IR of SU(1/2):



(7.14)

The second term corresponds to the trace. All these IR are atypical. They are connected together by odd generators of SU(1/2), some of them being lowering others raising. This can be schematized as follows:



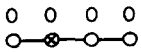
(7.15)

arrows representing odd generators. A similar example has been given by Scheunert, Nahm, and Rittenberg.[13]

Such a representation is said to be reducible (it contains an invariant subspace) but not decomposable. Another example is given by the supertableau of SU(2/3):

(7.16)

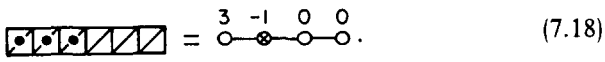
whose highest weight corresponds to the Kac–Dynkin diagram



(7.17)

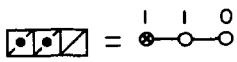
and hence is also reducible, although here $m+n<MN$.

On the other hand, the adjoint representation (7.1) is clearly irreducible, as well as the typical representation of SU(2/3):



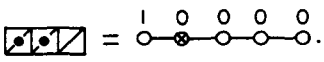
(7.18)

Also, the supertableau (7.7) corresponds to the typical representation of SU(1/3),



(7.19)

and to the atypical IR of SU(2/4)

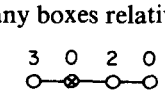


(7.20)

Thus, if M, N are sufficiently large, the mixed tableau is irreducible.

We can now address the following question: Can every IR of SU(M/N), as given by Kac–Dynkin diagram, be represented by a supertableau? We have already discussed in Sec. 6 the problem of typical representations, where one gets naturally integer values for a_M. Allowing the overall U(1) phase, it appears that we recover arbitrary values of a_M.

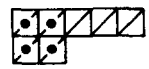
For atypical representations one has to consider Eq. (5.11) for covariant tensors, (5.20) for contravariant tensors, and combine them for mixed tensors. Hence, given the Kac–Dynkin labels for a_i, one has to solve for the supertableau labels b_i and $\bar{b}_i$. Clearly, there are several solutions and in most cases there is a supertableau corresponding to a Kac–Dynkin diagram. But again we somtimes face the difficulty of reducibility when the resulting solution contains too many boxes relative to M, N. For example, for SU(2/3),



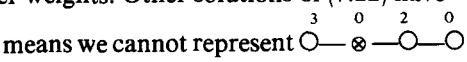
$$a_1=3,\quad a_2=0,\quad a_3=2,\quad a_4=0, \tag{7.21}$$

$$\begin{aligned}
&a_1=b_1-b_2+\bar{b}'_1-\bar{b}'_2=3,\\
&a_2=b_2+C'_1-\bar{b}'_1-\bar{C}_3=0,\\
&a_3=C'_1-C'_2+\bar{C}_2-\bar{C}_3=2,\\
&a_4=C'_2-C'_3+\bar{C}_1-\bar{C}_2=0,\\
&\bar{b}'_i=(\bar{b}_i-3)\theta(\bar{b}_i-3),\\
&C'_j=(C_j-2)\theta(C_j-2).
\end{aligned} \tag{7.22}$$

The solution with the minimal number of boxes is the supertableau:



(7.23)

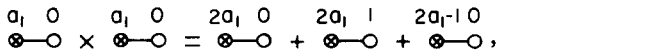
However, this is reducible and contains not only (7.21), but also IR with lower weights. Other solutions of (7.22) have more boxes. This means we cannot represent 

for SU(2/3) with an irreducible tensor.

In conclusion, for each atypical IR one can find a super Young tableau. Sometimes, this latter is reducible and contains also IR with lower weights (obtained by applying odd lowering operators).

8. TENSOR PRODUCTS OF IR

Scheunert, Nahm, and Rittenberg[13] have shown that the tensor product of IR of superalgebras is not always fully reducible. This is due to the fact, mentioned in Sec. 3, that atypical representations are not always fully reducible. The example they give is for SU(1/2):



$$\underline{4}\times\underline{4}=\underline{4}+\underline{8}+\underline{4} \tag{8.1}$$

($a_1\neq 0, 1$); (8.1) has been obtained by explicitly constructing all sixteen states.

There is a problem for $a_1=\frac{1}{2}$, because the representations (1, 0) and (0, 0) are typical, that is, of dimension 3, resp. 1. This means that they hide, in a nonreduced form, representations of dimension 1, resp. 3. It can be shown that the complete reduction is not possible.

Another example is shown in Sec. 7. Keeping this in mind, we can still try to learn from the rules of tensor products for classical Lie algebras, especially SU(M).

There are two main methods: Dynkin diagrams and Young tabeleaux. If the two IR to be multiplied have highest weights Λ_1 and Λ_2, the decomposition of the product contains the maximal highest weight

$$\Lambda_{\max}=\Lambda_1+\Lambda_2. \tag{8.2}$$

The next to the maximal Λ is obtained by the method of minimal chain.[14] By definition a sequence of simple roots $\alpha_{i_1}, \alpha_{i_2},\ldots,\alpha_{i_k}$ is a minimal chain linking Λ_1 and Λ_2 if the following two requirements are fulfilled: (1) $(\Lambda_1, \alpha_{i_1})\neq 0$, $(\alpha_{i_1}, \alpha_{i_2})\neq 0,\ldots,(\alpha_{i_k}, \Lambda_2)\neq 0$ and (2) no simple root can be removed from the sequence without violating (1). One now gets the highest weight of an IR contained in the decomposition of the product by subtracting the minimal chain from $\Lambda_{\max}$:

$$\Lambda = \Lambda_1 + \Lambda_2 - \sum_{j=i_1}^{i_k} \alpha_i . \tag{8.3}$$

(8.2) obviously gives the right result for the product (8.1):

$$\Lambda_{\max} = (a_1, 0) + (a_1, 0) = (2a_1, 0). \tag{8.4}$$

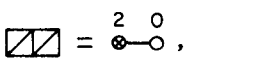
Caution is needed to apply (8.3), since $(\alpha_M, \alpha_M) = 0$. So we modify the definition: α_M can be subtracted from a weight, that is, belongs, to the minimal chain, if that weight has a nonzero M th component and was not obtained itself by subtracting α_M.

For SU(1/2), $\alpha_M = \alpha_1 = (0, -1)$, and

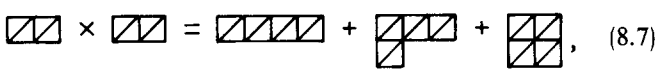
$$\Lambda = \Lambda_{\max} - \alpha_1 = (2a_1, 1), \tag{8.5}$$

which is the second term in (8.1). The third term is obtained by orthogonality.

Similarly, one may try to apply Young tableau techniques. Again, some changes are necessary. We have shown that supertableaux correspond to integer values of a_M, for atypical representations, as they should, but also for typical representations (modulo the phase, which restores arbitrary values of a_M). Actually, the product does not depend on the value a_M, as long as one stays with typical representations. Thus in the example (8.1)



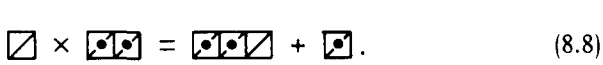
(8.6)


(8.7)

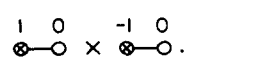
which, using (5.11), exactly agrees with (8.1) for $a_1 = 2$.

For pure covariant or pure contravariant tensors, we have found no example where the usual rules for Young tableaux do not apply.

For mixed tensors, the supertableau give again the correct result if M,N are sufficiently large. But for M,N small compared to the number of boxes the situation is complicated, since one encounters reducible representations. Still, the rules are useful. For example, for SU(1/2),


(8.8)

The left-hand side has dimension 3×4. From (7.10), we see that the right-hand side has also dimension 12. Working with Dynkin diagrams we see that the lhs is



(8.9)

Using the above rule, we see that it contains an IR with highest weight $\Lambda_1 + \Lambda_2 = (0, 0)$, and using the minimal chain, one with $\Lambda = \Lambda_1 + \Lambda_2 - \alpha_1 = (0, 1)$. Thus we indeed get two IR contained in (7.14).

The conclusion is that the usual rules seem to work, provided one decomposes the reducible representations, as we have shown above.

[1]V. G. Kac, Adv. Math. **26**(1), 8–96 (1977); V. G. Kac, Commun. Math. Phys. **53**, 31–64 (1977); V. G. Kac, Commun. Math. Phys. **53**, 31-64 (1977); V. G. Kac, *Lecture Notes in Mathematics* **676** (Springer-Verlag, New York, 1978), pp. 597-626.

[2]A. B. Balantekin and I. Bars, J. Math. Phys. **22**, 1149 (1981); **22**, 1810 (1981); **23**, 1239 (1982). See also P. H. Dondi and P. D. Jarvis, J. Phys. A **14**, 547 (1981).

[3]A. B. Balantekin, J. Math. Phys. **23**, 486 (1982).

[4]I. Bars, Preprint YTP-81-25, 1981 Yale University, in *Proceedings of School of Supersymmetry, Mexico, December 1981* (Plenum, New York) (to be published).

[5]This result was obtained independently by I. Bars, Ref. 4, and by B. Morel and H. Ruegg.

[6]J.-P. Hurni and B. Morel, University of Geneva, Preprints UGVA-DPT 1981/03-280 and UGVA-DPT 1981/11-322, 1981, J. Math. Phys. (to be published); F. Bordi, J.-P. Hurni, and B. Morel, UGVA-DPT 1982/01-332, University of Geneva, 1982.

[7]A. B. Balantekin, I. Bars, and F. Iachello, Phys. Rev. Lett. **47**, 19 (1981); Nucl. Phys. A **370**, 284 (1981); "New Supersymmetries in Nuclei," preprint, Yale University, March 1982.

[8]T. Banks, A. Schwimmer, and S. Yankielowicz, Phys. Lett. B **96**, 67 (1980). J. Thierry-Mieg and B. Morel, Harvard Preprint HUTMP 80/8100, 1980 Harvard University, and Preprint UGVA-DPT 1981/01-277, University of Geneva 1981; I. Bars and S. Yankielowicz, Phys. Lett. B **101**, 159 (1981); I. Bars, Phys. Lett. B **106**, 105 (1981); Preprint YTP 82-04, Yale University, 1982, to be published in Nucl. Phys.; YTP 82-10, Yale University, 1982; to be published in Phys. Lett.; YTP 82-12, Yale University, 1982, to be published in the Proceedings of the Rencontre de Moriond 1982; A. Schwimmer, Preprint, Yale University, 1982.

[9]A. B. Balantekin, Ph.D. thesis, Yale University, 1982.

[10]I. Bars and M. Günaydin, CERN Preprint TH 3350, 1982 (to appear in Commun. Math. Phys.).

[11]V. G. Kac, Ref. 1; V. G. Kac, Funct. Anal. Appl. **9** (3), 91 (1975); M. Scheunert, W. Nahm, and V. Rittenberg, J. Math. Phys. **17**, 1626–40 (1976); W. Nahm, V. Rittenberg, and M. Scheunert, Phys. Lett. B **61**, 383 (1976); P. G. O.. Freund and I. Kaplansky, J. Math. Phys. **17**, 228 (1976); G. Hochschild III, J. Math. **20**, 107 (1976); D. Z. Djokovic and G. Hochshild III, J. Math. **20**, 134 (1976); D. Z. Djokovic, J. Pure Appl. Alg. **7**, 217 (1976); Y. Ne'eman and S. Sternberg, Proc. Nat. Acad. Sci. USA **77**, 3127 (1980); P. H. Dondi and P. D. Jarvis, Z. Physik **14**, 201 (1980); M. Marcu, J. Math. Phys. **21**, 1277 (1980).

[12]R. C. King, J. Math. Phys. **11**, 280–93 (1970).

[13]M. Scheunert, W. Nahm, and V. Rittenberg, J. Math. Phys. **18**, 155–62 (1977).

[14]P. A. Rowlatt, *Lie Algebras* (Longmans Green, London, 1966); W. G. McKay and J. Patera, *Tables for Simple Lie Algebras* (Marcel Dekker, New York, 1981).

The SO(7) polynomial basis for symmetric representations

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A polynomial basis is derived for the symmetric irreducible representations of the group SO (7). The reduction of SO(7) into $[SU(2)]^3$ is considered. The SO(7) generators not belonging to $[SU(2)]^3$ are grouped into a bispinor vector, of which matrix elements are calculated. An explicit expression for the state vector is given.

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I. INTRODUCTION

In a previous article,[1] the reduced matrix elements of the generators of SO(7) have been calculated for symmetric irreducible representations. These reduced matrix elements followed as a solution of a set of induction equations. Here, we want to retrieve the same results in a completely different way. Starting from the basic [1,0,0] symmetric representation, a polynomial basis for SO(7) is constructed, analogous to that for the SO(5) group given by Sharp and Pieper.[2]

Not only are the reduced matrix elements, already previously derived,[1] obtained as a result, but with the present technique it also becomes possible to establish an explicit expression for the state vectors. By considering this specific method, and by applying in the near future an extension of the shift operator technique as used by Hughes[3] in his study of the SO(5)↓$[SU(2)]^2$ reduction, we hope to gain some more insight in the way the SO(7)↓$[SU(2)]^3$ state labeling problem for general representations can be solved.

II. THE CHOICE OF GENERATORS

The symmetric [1,0,0] representation of the SO(7) group decomposes into the $[SU(2)]^3$ representations[4] $(\frac{1}{2},\frac{1}{2},0)$ and (0,0,1). We denote the respective basis states by

$$\begin{aligned}
a &= |\tfrac{1}{2}\ \tfrac{1}{2}\ 0;\ \tfrac{1}{2}\ \tfrac{1}{2}\ 0\rangle, & b &= |\tfrac{1}{2}\ \tfrac{1}{2}\ 0;\ -\tfrac{1}{2}\ \tfrac{1}{2}\ 0\rangle,\\
c &= |\tfrac{1}{2}\ \tfrac{1}{2}\ 0;\ \tfrac{1}{2}\ -\tfrac{1}{2}\ 0\rangle, & d &= |\tfrac{1}{2}\ \tfrac{1}{2}\ 0;\ -\tfrac{1}{2}\ -\tfrac{1}{2}\ 0\rangle,\\
e &= |0\ 0\ 1;\ 0\ 0\ 1\rangle, & f &= |0\ 0\ 1;\ 0\ 0\ 0\rangle,\\
g &= |0\ 0\ 1;\ 0\ 0\ -1\rangle.
\end{aligned} \tag{2.1}$$

The familiar commutation relations for the generators of the $[SU(2)]^3$ subgroup algebra are obtained if we represent the generators by the following differential operators:

$$\begin{aligned}
&S_3 = \tfrac{1}{2}(a\partial_a - b\partial_b + c\partial_c - d\partial_d),\\
&T_3 = \tfrac{1}{2}(a\partial_a + b\partial_b - c\partial_c - d\partial_d),\\
&U_3 = e\partial_e - g\partial_g,\\
&S_+ = a\partial_b + c\partial_d, \quad S_- = b\partial_a + d\partial_c,\\
&T_+ = a\partial_c + b\partial_d, \quad T_- = c\partial_a + d\partial_b,\\
&U_+ = \sqrt{2}(e\partial_f + f\partial_g), \quad U_- = \sqrt{2}(f\partial_e + g\partial_f).
\end{aligned} \tag{2.2}$$

It is known that certain linear combinations of the remaining 12 generators form a bispinor vector with respect to the $[SU(2)]^3$ subalgebra.[1] It can be verified that the following operators have this property:

$$\begin{aligned}
&T^{[1/2\ \ 1/2\ \ 1]}_{1/2\ \ 1/2\ \ 0} = (1/\sqrt{2})(a\partial_f - f\partial_d), && T^{[1/2\ \ 1/2\ \ 1]}_{-1/2\ \ -1/2\ \ 0} = -(1/\sqrt{2})(f\partial_a - d\partial_f),\\
&T^{[1/2\ \ -1/2\ \ 1]}_{1/2\ \ -1/2\ \ 0} = (1/\sqrt{2})(c\partial_f + f\partial_b), && T^{[1/2\ \ 1/2\ \ 1]}_{-1/2\ \ 1/2\ \ 0} = (1/\sqrt{2})(f\partial_c + b\partial_f),\\
&T^{[1/2\ \ 1/2\ \ 1]}_{1/2\ \ 1/2\ \ 1} = -(1/\sqrt{2})(a\partial_g + e\partial_d), && T^{[1/2\ \ 1/2\ \ 1]}_{-1/2\ \ -1/2\ \ -1} = -(1/\sqrt{2})(g\partial_a + d\partial_e),\\
&T^{[1/2\ \ -1/2\ \ 1]}_{1/2\ \ -1/2\ \ 1} = (1/\sqrt{2})(e\partial_b - c\partial_g), && T^{[-1/2\ \ 1/2\ \ 1]}_{-1/2\ \ 1/2\ \ -1} = -(1/\sqrt{2}(b\partial_e - g\partial_c),\\
&T^{[-1/2\ \ 1/2\ \ 1]}_{-1/2\ \ 1/2\ \ 1} = (1/\sqrt{2})(e\partial_c - b\partial_g), && T^{[1/2\ \ -1/2\ \ 1]}_{1/2\ \ -1/2\ \ -1} = -(1/\sqrt{2})(c\partial_e - g\partial_b),\\
&T^{[-1/2\ \ -1/2\ \ 1]}_{-1/2\ \ -1/2\ \ 1} = -(1/\sqrt{2})(e\partial_a + d\partial_g), && T^{[1/2\ \ 1/2\ \ 1]}_{1/2\ \ 1/2\ \ -1} = -(1/\sqrt{2}(a\partial_e + g\partial_d).
\end{aligned} \tag{2.3}$$

Besides, this choice reproduces the correct commutator relations (2.28) reported by Vanden Berghe *et al.*[1]

Following the reasoning of Sharp and Pieper,[2] state vectors of an SO(7) symmetric irrep can be described by homogeneous polynomials in the basis states (2.1). The SO(7) scalar of second degree is proportional to $ad - bc - eg + \frac{1}{2}f^2$. The number

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of monomials of degree v in seven variables is $\binom{v+6}{6}$. In order to avoid duplication from that number of states in the $[v,0,0]$ representation, those which are proportional to powers of the scalar, i.e., $\binom{v+4}{6}$ must be subtracted. We find a number[5] which corresponds exactly to the dimension of the representation $[v,0,0]$ of SO(7), namely, $\binom{v+5}{5}+\binom{v+4}{5}$.

III. CALCULATION OF THE STATE VECTOR

In conformity with a previously introduced notation,[1] the state vectors will be denoted by $|su,\lambda\mu\nu\rangle$. A state corresponding to maximum angular momentum projections is necessarily of the form

$$|su,ssu\rangle = N_{su}a^{2s}e^{u}\sum_{x} f(x) \times(-eg+\tfrac{1}{2}f^2)^{(v-2s-u-2x)/2}(ad-bc)^x.$$

Indeed, the correct $[SU(2)]^3$ behavior is reproduced because according to our choice of generators, $ad-bc$ and $-eg+\frac{1}{2}f^2$ are the only basic $[SU(2)]^3$ scalars. Also, the state contained in SO(7) is a polynomial of degree v in a,b,c,d,e,f.

Because $\partial_a\partial_d-\partial_b\partial_c-\partial_e\partial_g+\frac{1}{2}\partial_f^2$ commutes with all the generators of SO(7) and annihilates the perimeter states $a^v/\sqrt{v!}$ and $e^v/\sqrt{v!}$, we have

$$(\partial_a\partial_d-\partial_b\partial_c-\partial_e\partial_g+\tfrac{1}{2}\partial_f^2)|su,ssu\rangle=0.$$

This property determines $f(x)$ except for a multiplicative constant and we find

$$|su,ssu\rangle = N_{su}a^{2s}e^{u}\sum_{x}\frac{((v-2s+u-2x)/2)!(-eg+\frac{1}{2}f^2)^{(v-2s-u-2x)/2}(bc-ad)^x}{4^x x!(v-2s+u-2x+1)!((v-2s-u-2x)/2)!(x+2s+1)!}. \tag{3.1}$$

In order to obtain an explicit expression for N_{su} we calculate now the right- and left-hand sides of

$$T^{[1/2\ 1/2\ 1]}_{1/2\ 1/2\ 1}|su,ssu\rangle = |s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle\langle s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1|T^{[1/2\ 1/2\ 1]}_{1/2\ 1/2\ 1}|su,ssu\rangle$$

and

$$\begin{aligned}T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ -1}&|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle\\ &=|s+1u+2,ssu\rangle\langle s+1u+2,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ -1}|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle\\ &+|s+1u,ssu\rangle\langle s+1u,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ -1}|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle\\ &+|su+2,ssu\rangle\langle su+2,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ -1}|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle\\ &+|su,ssu\rangle\langle su,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ -1}|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle.\end{aligned}$$

By equating successively the coefficients of $a^{2s+1}e^{u+1}(-eg+\frac{1}{2}f^2)^{(v-2s-u-2)/2}$ in the first equation and $(-eg)^{(v-2s-u)/2}$ and $(\frac{1}{2}f^2)^{(v-2s-u)/2}$ in the second equation, and taking into account that

$$\langle s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1|T^{[1/2\ 1/2\ 1]}_{1/2\ 1/2\ 1}|su,ssu\rangle=\langle su,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ -1}|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1\rangle,$$

we can prove that

$$\frac{N_{s+(1/2)u+1}}{N_{su}}=\epsilon\left[\frac{(v+2s+u+5)(2s+2)(2u+3)}{(v-2s-u)(2s+1)(2u+2)}\right]^{1/2},\quad \epsilon=1\ \text{ or }\ -1, \tag{3.2}$$

and similarly that

$$\langle s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u+1|T^{[1/2\ 1/2\ 1]}_{1/2\ 1/2\ 1}|su,ssu\rangle=\epsilon\left[(v-2s-u)(v+2s+u+5)\frac{(2s+1)(2u+2)}{8(2s+2)(2u+3)}\right]^{1/2}. \tag{3.3}$$

A second recursion relation for the normalization coefficient is obtained by considering the equalities

$$\begin{aligned}T^{[1/2\ 1/2\ \ 1]}_{1/2\ 1/2\ -1}|su,ssu\rangle&=|s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1\rangle\langle s+\tfrac{1}{2}u+1,s+\tfrac{1}{2}u-1|T^{[1/2\ 1/2\ \ 1]}_{1/2\ 1/2\ -1}|su,ssu\rangle\\ &+|s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1\rangle\langle s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1|T^{[1/2\ 1/2\ \ 1]}_{1/2\ 1/2\ -1}|su,ssu\rangle\end{aligned}$$

and

$$\begin{aligned}T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ 1}&|s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1\rangle\\ &=|s+1u,ssu\rangle\langle s+1u,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ 1}|s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1\rangle\\ &+|su,ssu\rangle\langle su,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ 1}|s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1\rangle.\end{aligned}$$

Here, the coefficients of $(bc)^{(v-2s-u)/2}$ in the first equation and $(-eg)^{(v-2s-u)/2}$ in the second equation are successively equated. On taking into account that

$$\langle s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1|T^{[1/2\ 1/2\ \ 1]}_{1/2\ 1/2\ -1}|su,ssu\rangle=\langle su,ssu|T^{[\ 1/2\ \ 1/2\ \ 1]}_{-1/2\ -1/2\ 1}|s+\tfrac{1}{2}u-1,s+\tfrac{1}{2}s+\tfrac{1}{2}u-1\rangle,$$

straightforward calculations lead to

$$\frac{N_{s+\frac{1}{2}u-1}}{N_{su}}=\epsilon'\left[\frac{(v+2s-u+4)(2s+2)2u}{(v-2s+u+1)(2s+1)16(2u+1)}\right]^{1/2}\epsilon'=1\text{ or }-1, \tag{3.4}$$

$$\langle s+\tfrac{1}{2}u-1,\, s+\tfrac{1}{2}s+\tfrac{1}{2}u-1|T^{\begin{bmatrix}1/2 & 1/2 & 1\\ 1/2 & 1/2 & -1\end{bmatrix}}|su,ssu\rangle = -\epsilon'\left[(v-2s+u+1)(v+2s-u+4)\frac{(2s+1)2u}{8(2s+2)(2u+1)}\right]^{1/2}. \tag{3.5}$$

The reduced matrix elements can be calculated by means of the Wigner–Eckart theorem applied to the $[SU(2)]^3$ state basis, namely,

$$\langle s'u',\lambda'\mu'\nu'|T^{\begin{bmatrix}1/2 & 1/2 & 1\\ \alpha & \beta & \gamma\end{bmatrix}}|su,\lambda\mu\nu\rangle = \langle s'u'\|T^{\begin{bmatrix}1/2 & 1/2 & 1\\ \alpha & \beta & \gamma\end{bmatrix}}\|su\rangle(2s'+1)^{-1}(2u'+1)^{-1/2}$$
$$\times\langle s\lambda\,\tfrac{1}{2}\alpha|s'\lambda'\rangle\langle s\mu\tfrac{1}{2}\beta\,|s'\mu'\rangle\langle u\nu 1\gamma|u'\nu'\rangle.$$

One finds

$$\langle s+\tfrac{1}{2}u+1\|T^{[1/2\ 1/2\ 1]}\|su\rangle = \epsilon\tfrac{1}{2}[(v-2s-u)(v+2s+u+5)(2s+1)(2s+2)(u+1)]^{1/2},$$
$$\langle s+\tfrac{1}{2}u-1\|T^{[1/2\ 1/2\ 1]}\|su\rangle = -\epsilon'\tfrac{1}{2}[(v-2s+u+1)(v+2s-u+4)(2s+1)(2s+2)u]^{1/2},$$

whereby ϵ and ϵ' are the same sign factors as in Eqs. (3.2)–(3.5).

It is known that for the perimeter state,

$$|0v,00v\rangle = e^v/\sqrt{v!}.$$

But according to (3.1), $|0v,00v\rangle = N_{0v}e^v\, v!/(2v+1)!$. Hence $N_{0v} = (2v+1)!/(v!\sqrt{v!})$. Together with the recursion relations (3.2) and (3.4), this is all we need in order to establish N_{su} in closed form. The final result reads

$$N_{su} = \epsilon^{(v-2s-u)/2}\epsilon'^{(v+2s-u)/2}2^{(v-2s-u)/2}/u!$$
$$\times\left[(2s+1)\frac{((v+2s-u+2)/2)!((v-2s-u/2))!(v+2s+u+3)!(v-2s+u+1)!(2u+1)!(v+1)!}{((v+2s+u+2)/2)!((v-2s+u)/2)!(2v+3)!}\right]^{1/2} \tag{3.6}$$

Since the choice of the phase factors and ϵ and ϵ' is still free, we can set $\epsilon = -\epsilon' = 1$. Precisely this choice makes the expressions (3.3) and (3.5) coincide with our previous results.[1]

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[1]G. Vanden Berghe, H. De Meyer, and P. De Wilde, J. Phys. A: Math. Gen. **15**, 2677 (1982).

[2]R. T. Sharp and S. C. Pieper, J. Math. Phys. **9**, 663 (1968).

[3]J. W. B. Hughes, Physica A **114**, 289 (1982).

[4]W. G. McKay and J. Patera, *Table of dimensions, indices and branching rules for representations of simple Lie algebras* (Marcel Dekker, New York, 1981).

[5]H. De Meyer, P. De Wilde, and G. Vanden Berghe, J. Phys. A: Math. Gen. **15**, 2665 (1982).

A new approach to permutation group representations. II

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The formula for the Yamanouchi matrix elements is rederived by the eigenfunction method in a simple fashion.

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I. INTRODUCTION

In Ref. 1 a new approach to the study of group representations is proposed, in which the calculation of the primitive characters, the irreducible bases, the Clebsch–Gordan coefficients, the isoscalar factors, etc., are all reduced to a single recipe—seeking the eigenfunctions of a certain kind of complete set of commuting operators (CSCO). It is in consequence termed the eigenfunction method. The structure of the CSCO is very simple, being a linear combination of the two-cycle class operators of the permutation groups. The new approach is self-contained, while the eigenfunction method has the advantage of being easily programmable and has wide applications in: the calculations of the Clebsch–Gordan coefficients and the outer-product reduction coefficients[2]– the tranformation matrices from the Yamanouchi basis of $S(n)$ to the $S(n) \supset S(n_1)\times S(n_2)$ basis[3], the coefficients of fractional parentage for the $SU(mn)\supset SU(m)\times SU(n)$ basis[4,5]; and the $SU(m+n)\supset SU(m)\times SU(n)$ basis,[6] etc.

However, an important problem is left over in Ref. 1, i.e., the independent derivation of the formula for the Yamanouchi matrix elements by the eigenfunction method. In this short paper we supply this derivation, which turns out to be the shortest way to reach this elegant formula.

II. EIGENVALUES OF THE TWO-CYCLE CLASS OPERATORS[7]

Let $|Y_1^{(\nu)}\rangle$ be the Yamanouchi basis vector having the maximum Yamanouchi numbers in the irreducible representation (irrep) $(\nu)=(\nu_1\nu_2\cdots)$, i.e., the Young tableau $Y_1^{(\nu)}$ has numbers 1 to ν_1 in the first row, ν_1+1 to $\nu_1+\nu_2$ in the second row, etc. We split the two-cycle class operator C_n into three parts:

$$C_n=\sum_{j>i=1}^{n}(ij)=T_r+T_c+T_m, \tag{1}$$

where T_r (T_c) is the sum of the transpositions i and j which are in the same rows (columns) of $Y_1^{(\nu)}$, and T_m is the remaining part of C_n. Let A be the product of the antisymmetrizers for the columns of $Y_1^{(\nu)}$. From

$$[C_n, A]=0 \tag{2a}$$

and

$$[T_c, A]=0, \tag{2b}$$

we have

$$[T_r+T_m, A]=0. \tag{2c}$$

Therefore

$$\begin{aligned} C_nA|Y_1^{(\nu)}\rangle &= AT_r|Y_1^{(\nu)}\rangle + T_cA|Y_1^{(\nu)}\rangle + AT_m|Y_1^{(\nu)}\rangle \\ &= \left[\frac{1}{2}\sum_i \nu_i(\nu_i-1)-\frac{1}{2}\sum_i \mu_i(\mu_i-1)\right] \\ &\quad \times A|Y_1^{(\nu)}\rangle + AT_m|Y_1^{(\nu)}\rangle, \end{aligned} \tag{3}$$

where $(\mu_1\mu_2\cdots)\equiv(\tilde{\nu})$ is the partition conjugate to (ν). In the following, we are going to show that $AT_m|Y_1^{(\nu)}\rangle$ as is identically zero.

Suppose i and j are the two numbers in the same row of $Y_1^{(\nu)}$, and i and k are the two numbers in the same coloumn of $Y_1^{(\nu)}$. According to

$$\begin{array}{ccccccc} i\ldots j & \overset{(ij)}{\rightarrow} & j\ldots i & \overset{(jk)}{\rightarrow} & k\ldots i & \overset{(ik)}{\rightarrow} & i\ldots k \\ \vdots & & \vdots & & \vdots & & \vdots \\ k & & k & & j & & j \end{array}$$

we know that

$$(jk)=(ik)(jk)(ij), \tag{4a}$$

where (ij) belongs to T_r and (ik) belongs to T_c. Consequently, we have

$$\begin{aligned} A(jk)|Y_1^{(\nu)}\rangle &= A(ik)(jk)(ij)|Y_1^{(\nu)}\rangle \\ &= A(ik)(jk)|Y_1^{(\nu)}\rangle = -A(jk)|Y_1^{(\nu)}\rangle, \end{aligned} \tag{4b}$$

where the property $A(ik)=-A$ has been used. Thus we proved that

$$A(jk)|Y_1^{(\nu)}\rangle=0, \tag{4c}$$

which implies

$$AT_m|Y_1^{(\nu)}\rangle=0. \tag{4d}$$

From Eqs. (3) and (4d) we have

$$C_nA|Y_1^{(\nu)}\rangle=\lambda_nA|Y_1^{(\nu)}\rangle, \tag{5a}$$

with the eigenvalue

$$\begin{aligned} \lambda_n &= \frac{1}{2}\sum_i \nu_i(\nu_i-1)-\frac{1}{2}\sum_i \mu_i(\mu_i-1) \\ &= \frac{1}{2}\sum_i \nu_i(\nu_i-1)-\frac{1}{2}\sum_l (\nu_l-\nu_{l+1})l(l-1) \\ &= \frac{1}{2}\sum_l \nu_l(\nu_l-2l+1). \end{aligned} \tag{5b}$$

From Eq. (2a) we know that the eigenvalues λ_n of C_n only depend on the irrep label (ν). Therefore Eq. (5a) is equivalent to

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$$C_n|Y_i^{(\nu)}\rangle = \lambda_n|Y_i^{(\nu)}\rangle, \tag{5c}$$

where $Y_i^{(\nu)}$ stands for any Young tableau of the irrep (ν).

III. DERIVATION OF THE YAMANOUCHI MATRIX ELEMENT

In Ref. 1 it was proved that a Yamanouchi basis vector of the permutation group S(n) is the simultaneous eigenvector of the $(n-1)$ two-cycle operators C_n, C_{n-1},... and C_2, and we can use the eigenvalues $(\lambda_n\lambda_{n-1}\cdots\lambda_2)$ of these operators to label a Yamanouchi basis vector, denoted as

$$|\lambda\rangle = |\lambda_n\lambda_{n-1}\cdots\lambda_2\rangle.$$

From the relations

$$[(n-1,n),C_{n-1}]\neq 0, \tag{6a}$$

$$[(n-1,n),C_f] = 0 \quad \text{for } f = n,n-2,n-3,\ldots,2, \tag{6b}$$

we know that the permutation $(n-1,n)$ has nonzero matrix elements only between the vectors $|\lambda_n\lambda'_{n-1}\lambda_{n-2}\cdots\lambda_2\rangle$ and $|\lambda_n\lambda_{n-1}\lambda_{n-2}\cdots\lambda_2\rangle$, i.e.,

$$\langle\lambda'_n\lambda'_{n-1}\lambda'_{n-2}\cdots\lambda'_2|(n-1,n)|\lambda_n\lambda_{n-1}\lambda_{n-2}\cdots\lambda_2\rangle = \text{const}\,\delta_{\lambda'_n\lambda_n}\delta_{\lambda'_{n-1}\lambda_{n-1}}\cdots\delta_{\lambda'_2\lambda_2}. \tag{7}$$

The following identity relations are easily established:

$$C_n = C_{n-1} + \sum_{i=1}^{n-1}(i,n), \tag{8a}$$

$$\sum_{i=1}^{n-2}(i,n-1) = C_{n-1} - C_{n-2}, \tag{8b}$$

$$C_n = C_{n-1} + (n-1,n)\sum_{i=1}^{n-2}(i,n-1)(n-1,n) + (n-1,n). \tag{8c}$$

Using Eqs. (8b) and (6a), Eq. (8c) becomes

$$C_n = C_{n-1} - C_{n-2} + (n-1,n)C_{n-1}(n-1,n) + (n-1,n). \tag{9a}$$

Written in a more elegant form, Eq. (9a) becomes

$$[C_{n-1},(n-1,n)]_+ = (C_n + C_{n-2})(n-1,n) - 1, \tag{9b}$$

where $[A,B]_+ = AB + BA$.

Inserting Eq.(9b) between the two Yamanouchi basis vectors $|\lambda'\rangle$ and $|\lambda\rangle$, and using Eqs. (5c) and (7), we obtain

$$\langle\lambda'_n\lambda'_{n-1}\cdots\lambda'_2|(n-1,n)|\lambda_n\lambda_{n-1}\cdots\lambda_2\rangle = \mu^{-1}\delta_{\lambda'_n\lambda_n}\delta_{\lambda'_{n-1}\lambda_{n-1}}\cdots\delta_{\lambda'_2\lambda_2},$$

$$\mu = \lambda_n - \lambda'_{n-1} - \lambda_{n-1} + \lambda_{n+2}. \tag{10}$$

From Eq. (10) we obtain the diagonal matrix element of the permutation $(n-1,n)$

$$\langle\lambda_n\lambda_{n-1}\cdots\lambda_2|(n-1,n)|\lambda_n\lambda_{n-1}\cdots\lambda_2\rangle = \sigma^{-1}, \tag{11a}$$

$$\sigma = \lambda_n - 2\lambda_{n-1} + \lambda_{n-2}, \tag{11b}$$

as well as the off-diagonal matrix elements

$$\langle\lambda_n\lambda'_{n-1}\cdots\lambda_2|(n-1,n)|\lambda_n\lambda_{n-1}\cdots\lambda_2\rangle = \begin{cases} 0 & \text{for } \mu\neq 0, \\ b & \text{for } \mu = 0, \end{cases} \tag{12}$$

where the coefficient b is to be determined.

Before determining the constant b, we first examine the conditions under which the permutation $(n-1,n)$ has nonzero off-diagonal matrix elements.

Let $Y_{\lambda'}$ and Y_λ be the two Young tableaux corresponding to the basis vectors $|\lambda_n\lambda'_{n-1}\lambda_{n-2}\cdots\lambda_2\rangle$ and $|\lambda_n\lambda_{n-1}\lambda_{n-2}\cdots\lambda_2\rangle$, which differ only in the second quantum numbers. Clearly, $Y_{\lambda'}$ and Y_λ must have the same Young diagrams (ν) and (ν'') for the numbers (1,2,...,n) and (1,2,...,$n-2$), respectively. If the numbers n and $n-1$ occupy the same row or column of the Young tableau Y_λ, the aforesaid condition implies that $Y_{\lambda'}$ and Y_λ must be identical; otherwise stated, $(n-1,n)$ does not have nonzero off-diagonal matrix elements between any two vectors $|\lambda'\rangle$ and $|\lambda\rangle$ if one of them is symmetric or antisymmetric in the indices $n-1$ and n. On the other hand, if n and $n-1$ are not in the same row or column of Y_λ, then the above condition means that $Y_{\lambda'}$ and Y_λ differ only in the interchange of the positions of n and $n-1$.

All taken together, only when n and $n-1$ are not in the same row or column of a Young tableau Y_λ, and also only for a unique Young tableau $Y_{\lambda'} = (n-1,n)Y_\lambda$, does the off-diagonal matrix element $\langle\lambda'|(n-1,n)|\lambda\rangle$ differ from zero.

According to the identity $(n-1,n)^2 = 1$, and the above discussion, we immediately have

$$\langle\lambda|(n-1,n)|\lambda\rangle^2 + \langle\lambda|(n-1,n)|\lambda'\rangle \times\langle\lambda'|(n-1,n)|\lambda\rangle = 1. \tag{13}$$

Consequently, the constant b is determined by the equation

$$|b|^2 = 1 - 1/\sigma^2. \tag{14a}$$

Under the Yamanouchi phase convention, we have

$$\langle\lambda'|(n-1,n)|\lambda\rangle = (\sigma^2-1)^{1/2}/|\sigma|. \tag{14b}$$

Finally, we show that the constant σ defined by Eq. (11b) is exactly the axial distance from n to $n-1$ in the Young tableau Y_λ.

Suppose the row numbers of the indices n and $n-1$ in the Young tableau Y_λ are l and l', respectively. It follows from Eqs. (11b) and (5b) that

$$\sigma = \tfrac{1}{2}[(f_l + f_{l'}) - 2(f_{l-1} + f_{l'}) + (f_{l-1} + f_{l'-1})] = \tfrac{1}{2}[(f_l - f_{l-1}) - (f_{l'} - f_{l'-1})], \tag{15}$$

where

$$f_l = \nu_l(\nu_l - 2l).$$

Therefore

$$\sigma = (\nu_l - l) - (\nu_{l'} - l'). \tag{16}$$

Equation (16) is precisely the expression for the axial distance given in Ref. 8.

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[1] J.-Q. Chen and M.-J. Gao, J. Math. Phys. **23**, 928 (1982).

[2] J.-Q. Chen and M.-J. Gao, *Reduction Coefficients of Permutation Groups and Their Applications* (Beijing Science, Beijing, 1981).

[3] J.-Q. Chen, D. Collinson, and M.-J. Gao, J. Math. Phys. (to be published).

[4] J.-Q. Chen, J. Math. Phys. **22**, 1 (1981).

[5] J.-Q. Chen *et al.*, SU(mn)⊃SU(m)×SU(n) single-particle fractional parentage coefficients for arbitrary m and n: *Algorithm and Tables* (to be published).

[6] J.-Q. Chen (unpublished).

[7] Here we followed the clue suggested by: J. P. Elliott and P. G. Dawber, *Symmetry in Physics* (MacMillan, London, 1979), Vol. 2.

[8] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. I.

The geometric theory of deformation and linearization of Pfaffian systems and its applications to system theory and mathematical physics. I[a)]

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Many physical and engineering problems lead to the study of the qualitative and geometric properties of differential equations and their solutions as a function of exogeneous parameters. The theory of *deformation of geometric structures and pseudogroups* initiated by K. Kodaira and D. C. Spencer deals with the common mathematical structure underlying these problems. This series of papers will adapt the work in the pure mathematics literature to the needs of the applications, with the emphasis on the theory of *deformations of Pfaffian systems*. The applied area to be emphasized in this first part is the theory of nonlinear input–output systems. I will also present the abstract algebraic structure which seems to underlie the theory of Pfaffian systems, which I call *Cartan–Vessiot filtrations of Lie algebras*.

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1. INTRODUCTION

Many problems of science and engineering involve differential equations depending on parameters. The name *bifurcation or perturbation theory* is often given to this subject. It has been extensively studied from both a geometric[1] and analytic[2] point of view. My aim here is to study it from the point of view of the Kodaira–Spencer theory of deformations of geometric structures.[3] Thus, I hope to develop the "applied" aspects of the Kodaira–Spencer deformation theory.

Let us illustrate with a traditional example of a singular perturbation. Consider a typical singular perturbation problem of mathematical physics, e.g., an ordinary nonlinear differential equation

$$\epsilon \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right), \tag{1.1}$$

depending on a parameter $\epsilon \neq 0$. Introduce the space of variables:

$$Z = \{(x,x',t)\}$$

[the space of 1-jets $J^1(R,R)$ of mappings $R \to R$].

Introduce the following 1-differential forms on Z:

$$\theta = dx - x'\,dt, \tag{1.2}$$

$$\omega_\epsilon = \epsilon\,dx' - f(x,x')\,dt. \tag{1.3}$$

For fixed $\epsilon \neq 0$, let $\mathscr{E}_\epsilon$ be the exterior differential system[4] generated by θ and ω_ϵ, i.e., the smallest algebraic ideal in the Grassmann algebra $\mathscr{D}(M)$ of smooth differential forms on Z which contains θ and ω_ϵ and is closed under the exterior derivative operation d. We can then regard a solution of (2.1) as a map

$$\sigma\colon R \to Z$$

such that the following conditions are satisfied:

$$\sigma^*(\mathscr{E}_\epsilon) = 0, \tag{1.4}$$

$$\sigma^*(dt) \neq 0. \tag{1.5}$$

Let $\mathscr{E}_0$ be the exterior differential system generated by the following 1-forms:

$$\theta = dx - x'\,dt, \tag{1.6}$$

$$\omega_0 = f(x,x')\,dt. \tag{1.7}$$

Let $\mathscr{E}_0'$ be the exterior differential system generated by the following 1-forms:

$$\theta = dx - x'\,dt, \quad \omega_0' = dt. \tag{1.8}$$

Finally, let $\mathscr{E}_0''$ be the system generated by the following 1-forms:

$$\theta = dx - x'\,dt, \quad \omega_0'' = f(x,x'). \tag{1.9}$$

We see that $\mathscr{E}_0$ is *decomposable*, in the obvious sense, into $\mathscr{E}_0'$ and $\mathscr{E}_0''$. Further, as $\epsilon \to 0$, $\mathscr{E}_\epsilon$ goes over perfectly smoothly [as an ideal in the Grassmann algebra $\mathscr{D}(Z)$] to $\mathscr{E}_0$.

Theorem 1.1: For $\epsilon \neq 0$, $\mathscr{E}_\epsilon$ is locally equivalent to the exterior differential system $\mathscr{E}$ generated by the following 1-forms:

$$\theta = dx - x'\,dt, \quad \omega'' = dx'. \tag{1.10}$$

Proof: "Local equivalence" means that each point p_0 of M has a neighborhood U and a diffeomorphism

$$\phi_\epsilon\colon U \to U$$

such that

$$\phi_\epsilon(\mathscr{E}) = \mathscr{E}_\epsilon. \tag{1.11}$$

To see how such a ϕ_ϵ may be found, it is most convenient to construct the dual vector field system [i.e., $\mathscr{F}(Z)$-submodule of $\mathscr{V}(Z)$] dual to the exterior systems $\mathscr{E}_\epsilon$ and $\mathscr{E}$. Construct the following vector fields:

$$V = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}, \tag{1.12}$$

$$V_\epsilon = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + \epsilon^{-1} f \frac{\partial}{\partial x'}. \tag{1.13}$$

Then,

$$\mathscr{E}(V) = 0 = \mathscr{E}_\epsilon(V'). \tag{1.14}$$

Now, there are (locally) diffeomorphisms

$$\phi\colon U \to M,$$

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where U is an open subset of M, such that

$$\phi_*(V) = V_\epsilon.$$

Because of relations (2.14), it is readily seen that ϕ carries $\mathscr{E}_\epsilon$ into $\mathscr{E}$, i.e., ϕ establishes the equivalence of $\mathscr{E}_\epsilon$, for $\epsilon \neq 0$, with the "canonical form" system $\mathscr{E}$.

We can now see what this means from the deformation-of-exterior differential system point of view:

$$\epsilon \to \mathscr{E}_\epsilon$$

is a family of exterior differential systems, which vary in a perfectly smooth way with ϵ. For $\epsilon_1, \epsilon_2 \neq 0$, $\mathscr{E}_{\epsilon_1}$ and $\mathscr{E}_{\epsilon_2}$ are *locally equivalent*. However, at $\epsilon = 0$ a singularity occurs: The system becomes decomposable. This situation is, from the general deformation-theoretic point of view reminiscent of the Inonu–Wigner "contraction-of-Lie-algebras" phenomenon,[5] namely:

A family $\epsilon \to \mathscr{L}_\epsilon$ of Lie algebras which are all isomorphic for $\epsilon \neq 0$, but with a change in algebraic structure at $\epsilon = 0$.

Let us now consider another traditional problem of mathematical physics which leads naturally to a deformation of a Pfaffian systems problem.

2. POINCARÉ LINEARIZATION OF VECTOR FIELDS WITH ZERO POINTS

Let

$$\frac{dx}{dt} = f(x), \quad x \in R^n, \tag{2.1}$$

be a nonlinear ordinary differential equation such that

$$f(0) = 0. \tag{2.2}$$

Suppose also that $x \to f(x)$ is a real analytic map of a neighborhood V of an R^n into R^n.

Poincaré introduced the technique of *linearization* for studying Eq. (2.2). He asked for conditions that there should exist analytic diffeomorphisms

$$\phi: \quad V \to V, \quad \phi(0) = 0$$

of a neighborhood U of zero in R^n such that ϕ carries the differential equation (2.1) over to a *linear* differential equation

$$\frac{dx}{dt} = Ax, \tag{2.3}$$

where A is an $n \times n$ matrix. His work has been extended in many directions in modern times, especially by Sternberg[6] and Chen.[7]

Poincaré's problem can be considered as a deformation-of-exterior-differential systems problem. Let Z be $R^n \times R$, the space of all pairs (x,t), $x \in R^n$, $t \in R$. Consider the 1-forms

$$dx - f\,dt$$

and the exterior differential system $\mathscr{E}$ they generate. [Since we are using vectorial notation, i.e., $x = (x_1,\ldots,x_n)$, there are n such forms.] The solutions of (2.1) are then the one-dimensional integral submanifolds of $\mathscr{E}$ such that

$$dt \neq 0.$$

For each $\epsilon \neq 0$, let

$$\phi_\epsilon: \ Z \to Z$$

be the maps defined as follows:

$$\phi_\epsilon(x,t) = (\epsilon x, t).$$

Then,

$$\phi_\epsilon^*(dx - f\,dt) = \epsilon\,dx - f(\epsilon x)\,dt.$$

Let $\mathscr{E}_\epsilon$ be the exterior differential system generated by the 1-forms

$$dx - \epsilon^{-1} f(\epsilon x)\,dt. \tag{2.4}$$

Then, we see that

$$\phi_\epsilon^*(\mathscr{E}) = \mathscr{E}_\epsilon. \tag{2.5}$$

Now, using the crucial hypothesis (2.2), we see that the deformation $\epsilon \to \mathscr{E}_\epsilon$ is smooth *also* at $\epsilon = 0$. If the Taylor expansion of f at $x = 0$ is

$$f(x) = Ax + A_2x^2 + \cdots,$$

then $\mathscr{E}_0$ is generated by the following 1-forms:

$$dx - Ax\,dt. \tag{2.6}$$

Linearization of (2.1), in the sense of Poincaré, means showing that the family

$$\epsilon \to \mathscr{E}_\epsilon$$

of exterior systems is equivalent, within the group of all diffeomorphisms of R^n which leave the origin fixed, to the *constant* family of systems

$$\epsilon \to \mathscr{E}_0.$$

In Refs. 8 and 9, I have shown how the Poincaré problem can be generalized (in one direction) and related to *Lie algebra cohomology*. In one such generalized case, Guillemin and Sternberg[10] have shown how the problem of linearization can be carried through in terms of *convergent* power series. Gel'fand and Fuks[11] have shown how certain of relevant Lie algebra cohomology groups can be computed in geometric terms. In a later paper in this series I plan to gather together this information to see what can be said in general about linearization of *completely integrable* (in the sense of Frobenius) systems. Rather than go immediately in this direction of generalizations, I will study the relation to the theory of nonlinear input–output systems.

Let us turn now to another example of interest and importance in control-system theory.[12,13]

3. DEFORMATION AND LINEARIZATION OF INPUT–OUTPUT SYSTEMS

We can now generalize the problem of linearization of a single differential equation (2.1) by considering *inputs* and *outputs*. Consider a set of differential equations of the following form:

$$\frac{dx}{dt} = f(x,u), \quad y = g(x), \tag{3.1}$$

$$x \in R^n, \quad u \in R^m, \quad y \in R^p.$$

In engineering terms, (3.1) leads to a "black box"

$$\text{—}^{u}\square\text{—}^{y}, \tag{3.2}$$

which accepts $t \to u(t)$, a curve in R^m, as *input* and calculates a curve $t \to y(t)$ in R^p as *output*. The output is then a solution of a time-dependent ordinary differential equation:

$$\frac{dx}{dt} = f(x(t),u(t)) \quad y = g(x(t)). \tag{3.3}$$

Another way (3.1) might appear in engineering applications is via "feedback" or "control" laws

$$x \to u(x), \tag{3.4}$$

which are maps: $R^n \to R^m$. Given (3.4), the output $t \to y(t)$ is found by solving the time-independent ordinary differential equations

$$\frac{dx}{dt} = f(x,u(x)), \tag{3.5}$$

with

$$y = g(x(t)).$$

Whatever the physical or engineering interrelation, a Pfaffian system constructed from (3.1)[13–15] plays a basic role. Let

$$Z = \text{space of } (x,u,y,t) \equiv R^n \times R^m \times R^p \times R. \tag{3.6}$$

Construct on Z the exterior differential system, generated by the following set of n 1-forms and p 0-forms:

$$\theta = dx - f(x,u)\,dt, \quad h = y - g. \tag{3.7}$$

The solution curves

$$t \to (x(t),u(t),y(t),t) \tag{3.8}$$

of $\mathscr{E}$ clearly correspond to the input–output relations (3.2)–(3.3).

One can now construct various pseudogroups on Z which, acting on $\mathscr{E}$, transform it into an $\mathscr{E}'$ arising from another input–output system. In this way, one obtains pseudogroups acting on space of systems. Precisely as we did in Sec. 2 for the single differential equation (2.1), we can then consider various deformation-equivalence problems.

Among the systems of type (3.3), the *linear* ones play a distinguished role, just as the linear differential equations play a distinguished role among the general class of differential equations of form (2.1). They are of the following form:

$$\frac{dx}{dt} = Ax - Bu, \quad y = Cx, \tag{3.9}$$

where A is an $n \times n$ matrix, B an $n \times m$ matrix, C a $p \times n$ matrix. The theory of this special class of systems is, of course, the most highly developed, and criteria for a nonlinear system of form (3.1) to be equivalent to a linear one of form (3.9) [in the sense that their associated exterior differential systems (3.9) are equivalent[16,17]] would be very important from both a practical and theoretical point of view. Such criteria have been under extensive development in the system theory literature.[18–39] In Ref. 39 I have shown that sufficient conditions for linearization (and associated "canonical forms") could be obtained, in some simple cases, after changes of notation, from classical work of Goursat.[17]

In order to proceed as we did in Sec. 2, let us first notice that the set of exterior differential systems of the form

$$dx - (Ax + Bu)\,dt, \quad y - Cx \tag{3.10}$$

admits a dilation group

$$\phi_\epsilon\colon (x,u,y,t) \to (\epsilon x,\epsilon u,\epsilon y,t) \quad \text{for } \epsilon \in R - (0). \tag{3.11}$$

Let us then let this dilation group act on the exterior differential system associated by (3.9) with the general input–output system (3.1).

$$\phi_\epsilon^*(\theta) = \epsilon\,dx - f(\epsilon x,\epsilon u)\,dt,$$
$$\phi^*(h) = \epsilon y - g(\epsilon x). \tag{3.12}$$

$\phi_\epsilon^*(\mathscr{E})$ is then generated by the 1-forms and 0-forms

$$dx - \epsilon^{-1} f(\epsilon x,\epsilon u)\,dt, \quad y - \epsilon^{-1} g(\epsilon x). \tag{3.13}$$

Theorem 3.1: Suppose that

$$(x,u) \to f(x,u), \quad x \to g(x)$$

are C^∞ (real analytic) maps defined in some neighborhood of 0 in $R^n \times R^m$ and R^n. Suppose that the following condition is satisfied:

$$f(0,0) = 0, \quad g(0) = 0. \tag{3.14}$$

Then, the exterior differential system $\phi_\epsilon^*(\mathscr{E})$ (defined in some neighborhood of 0 in $R^n \times R^m \times R^p$) depends in a C^∞ (real analytic) way on ϵ at $\epsilon = 0$. In other words, $\epsilon \to \mathscr{E}_\epsilon$ is a smooth deformation (or family) of exterior differential systems about $\epsilon = 0$. The system $\mathscr{E}_0$ is *linear* in the form (3.9), so that local "triviality" of the deformation $\epsilon \to \mathscr{E}_\epsilon$ about $\epsilon = 0$ *implies* linearization of the system (3.11) in the sense meant in system theory.[19,22]

Proof: The proof uses only calculus. The maps

$$(\epsilon,x,u) \to \epsilon^{-1} f(\epsilon x,\epsilon u),$$
$$(\epsilon,x,y) \to y - \epsilon^{-1} g(\epsilon x)$$

are smooth (i.e., C^∞ or real analytic) locally about the point 0 if and only if conditions (3.14) are satisfied.

Remark: It is noteworthy that the condition (3.14) also appears naturally in the work in the system theory literature on linearization by feedback.

Having seen that linearization of nonlinear systems can, just as for the Poincaré problem and the generalization developed in Refs. 8 and 9, be interpreted as a deformation-theory-of-Pfaffian systems, I will now turn to develop some general geometric insights into deformation theory.

4. THE INTUITIVE GEOMETRIC AND LIE THEORETIC VIEWPOINT IN THE GENERAL THEORY OF DEFORMATIONS AND BIFURCATIONS

We will now give an overview of certain features of deformation theory as it has evolved from the work of Kodaira and Spencer,[3] Kuranishi, Frohlicher, Nijenhuis and Richardson[40], Gerstenhaber, Richardson,[41] Guillemin and Sternberg[42] and myself.[43] We shall discuss "submanifolds," "maps," "groups," and "Lie algebras of groups" without being precise about their dimensionability and/or degree of smoothness.

Some of these ideas can be discussed more precisely and/or rigorously in terms of the theory of categories and sheaves and the theory of analysis and geometry on infinite-dimensional manifolds, but I will not go into this here, at

least in this first paper.

Let M be a manifold (possibly infinite-dimensional), with p denoting a typical point of M, G a group,

$$(g,p) \to gp, \quad G \times M \to M$$

a transformation group action of G on M. This action defines an equivalence relation on M:

p is equivalent to p' if and only if there is a $g \in G$ such that

$$p' = gp. \tag{4.1}$$

The equivalence classes of M by this relation are called *orbits*. The set of equivalence classes is called the *orbit space* (or *quotient space*), denoted as

$$G \backslash M. \tag{4.2}$$

The map

$$\pi\colon\ M \to G \backslash M, \tag{4.3}$$

which assigns to $p \in M$ the orbit to which it belongs is called the *projection map*.

The manifold structure will assign to each point, $p \in M$, a real vector space, M_p called the *tangent space to M at p*. For each $g \in G$, the map

$$g\colon\ M \to M, \quad p \to gp \tag{4.4}$$

will have a *differential* which will be a linear map

$$g_*\colon\ M_p \to M_{gp}. \tag{4.5}$$

The collection of pairs

$$(p,v), \quad p \in M,\ v \in M_p, \tag{4.6}$$

will form a new manifold

$$T(M),$$

called the *tangent bundle*. The map (4.5) defines a transformation group action

$$G \times T(M) \to T(M) \tag{4.7}$$

of G on $T(M)$ called the *tangent vector prolongation of the given action*.

Consider an orbit

$$N = Gp \tag{4.8}$$

of G. Suppose it is a submanifold of M. For $p \in N$ its tangent space N_p will be a linear subspace of M_p.

Set

$$G^p = \{\, g \in G\colon gp = p \,\}. \tag{4.9}$$

G^p is a subgroup of G. Form the coset space G/G^p. The map

$$g \to gp \tag{4.10}$$

identifies G/G^p with N.

G^p is called the *isotropy subgroup of G at p*. For $g \in G^p$, the map g_* maps N_p into itself; hence the assignment $g \to g_*$ defines a linear representation of G^p on N_p, called the *linear isotropy group*.

The quotient vector space

$$M_p/N_p \equiv N_p^{\perp} \tag{4.11}$$

is called the *normal vector space* to the submanifold N. The set of ordered pairs

$$(p,v), \quad p \in N,\ v \in N_p^{\perp}, \tag{4.12}$$

is called the *normal vector bundle to N*, denoted as

$$N^{\perp}. \tag{4.13}$$

The linear action

$$g \to \phi_{g^*}, \quad g \in G^p,$$

passes to the quotient to define a linear representation

$$G^p \to (\text{linear maps } N_p^{\perp} \to N_p^{\perp}). \tag{4.14}$$

This linear representation plays a basic role in determining the geometric structure of the orbit space

$$G \backslash M. \tag{4.15}$$

In fact, notice that the assignment

$$p \to N_p^{\perp}$$

defines a generalized *tangent vector bundle* to the orbit space (4.14). Only rarely is this orbit space a "manifold," in the usual sense. Some notion of "generalized manifold" must be used. In the Ehresmann theory,[44] which I am basically following, this is done by using the "pseudogroup" idea, just as an ordinary manifold is the coset space of its group of diffeomorphisms, so a "generalized manifold" is some sort of "generalized coset space" associated with a "pseudogroup."

We can now state some general aims of deformation theory:

(1) Compute and/or parametrize the orbit space $G \backslash M$;

(2) For each orbit N, compute and/or parametrize the normal vector space $N_p^{\perp}$;

(3) Compute the action of G^p on $N_p^{\perp}$.

5. THE ACTION OF GROUPS OF DIFFEOMORPHISMS ON PFAFFIAN SYSTEMS

Let us now specialize the general situation described in Sec. 4. Let Z be a finite-dimensional, C^∞ paracompact manifold. Let $\mathscr{F}(Z)$ denote the commutative associative ring, under pointwise multiplication of C^∞, real-valued functions on Z. Let $\mathscr{D}^1(Z)$ denote C^∞, 1-differential forms on Z. They form a *module* over $\mathscr{F}(Z)$.

Definition: A *Pfaffian system* $\mathscr{P}$ on Z is defined to be a free submodule of $\mathscr{D}^1(Z)$. The rank of the module is called the rank of the Pfaffian system.

Let $M(m)$ be the space of all Pfaffian systems of rank m. Let G be a group of C^∞ diffeomorphisms of Z, i.e., a transformation group on Z.

Remark: Ultimately, we will need the generalization where G is a *pseudogroup*, in the sense of Ehresmann,[44] and $M(m)$ is replaced by the sheaf whose fiber over a given point is the space of germs of submodules of rank m, but for the sake of simplicity and comprehensibility for those who are not familiar with these elaborations of basic "calculus on manifolds" theory, I will keep to the simpler "global" notation.

The action of G on Z by diffeomorphisms defines a transformation group action of G on $M(m)$ in the following way. For $g \in G$, g^* is an invertible R-linear map:

$$g^*\colon\ \mathscr{D}^1(z) \to \mathscr{D}^1(z).$$

The inverse is, of course, just $g^{-1}{}^*$.

If $\mathscr{P}$ is a Pfaffian system, note that

$$g^{-1}*(\mathscr{P}) = \{\, g^{-1}*(\theta) : \theta \in \mathscr{P} \,\} \tag{5.1}$$

is also.

Definition: The transform of a $\mathscr{P} \in M(m)$ by a $g \in G$ is the system $g^{-1}*(\mathscr{P})$ given by formula (5.1). This defines a transformation group action

$$(g,\mathscr{P}) \to g(\mathscr{P}) \equiv g^{-1}*(\mathscr{P}), \quad G \times M(m) \to M(m)$$

of G on $M(m)$.

From now on, we shall consider the integer m as constant and denote $M(m)$ by M. We now regard M as a sort of infinite-dimensional manifold and attempt to describe the tangent and normal bundle to the orbit.

Definition: Given $\mathscr{P} \in M$, the tangent space to M at the "point" $\mathscr{P}$ is the following vector space:

$$M_p = \mathrm{Hom}_{\mathscr{F}(Z)}(\mathscr{P}, \mathscr{D}^1(Z)/\mathscr{P}), \tag{5.2}$$

where $\mathrm{Hom}_{\mathscr{F}(Z)}(\ ,\ )$ denotes the space of $\mathscr{F}(Z)$-linear homomorphisms between the indicated $\mathscr{F}(Z)$-modules.

Remark: Notice that differential geometric objects (e.g., tensor fields) may be modules over various rings in nontrivial and geometrically important ways. Thus, if

$$\pi\colon\ Z \to Y$$

is a smooth mapping, multiplication by $\pi^*(\mathscr{F}(Y))$ defines an $\mathscr{F}(Y)$-structure on the cross sections of vector bundles over Z.

The *tangent bundle* $T(M)$ to M is the space of ordered pairs

$$(\mathscr{P},\gamma), \tag{5.3}$$

where $\mathscr{P}$ is an element of M, i.e., a free submodule of $\mathscr{D}^1(Z)$ of rank m and γ is an $\mathscr{F}(Z)$-linear homomorphism from the module $\mathscr{P}$ to the quotient module $\mathscr{D}^1(Z)/\mathscr{P}$.

The action of the group G of diffeomorphisms on Z on $T(M)$ is then the natural linear action induced by $g^{-1}*$ for $g \in G$.

Let us suppose now that the G is generated by a Lie algebra $\mathscr{G}$ of vector fields on M. ($\mathscr{G}$ may be infinite-dimensional.) Thus, each $V \in \mathscr{G}$ is an element of $\mathscr{V}(Z)$, the Lie algebra derivation of $\mathscr{F}(Z)$. These derivations extend to the *Lie derivative operation* on differential forms:

$$\theta \to \mathscr{L}_V(\theta). \tag{5.4}$$

The Lie derivative operation is *not* $\mathscr{F}(Z)$-linear; instead we have

$$\mathscr{L}_V(f\theta) = V(f)\theta + f\mathscr{L}_V(\theta) \quad \text{for } f \in \mathscr{F}(Z). \tag{5.5}$$

Thus, if we take $\mathscr{P}$ to be an $\mathscr{F}(Z)$-submodule of $\mathscr{D}^1(Z)$, and define the mapping, for $\theta \in \mathscr{P}$,

$$\alpha_V(\theta) \equiv \mathscr{L}_V(\theta) \quad \text{projected modules } \mathscr{P} \text{ into } \mathscr{D}^1(Z)/\mathscr{P}, \tag{5.6}$$

we obtain an *$\mathscr{F}(Z)$-linear mapping*

$$\alpha_V\colon\ \mathscr{P} \to \mathscr{D}^1(Z)/\mathscr{P}. \tag{5.7}$$

In other words,

$$\alpha_V \in \mathrm{Hom}_{\mathscr{F}(Z)}(\mathscr{P}, \mathscr{D}^1(Z)/\mathscr{P}) \equiv M_p.$$

Now we can state a main result.

Theorem 5.1: Let $\mathscr{P}$ be an element of M, i.e., a Pfaffian system of rank m. The linear space

$$\alpha_V\colon\ V \in \mathscr{G} \tag{5.8}$$

is the tangent space $(G\mathscr{P})_{\mathscr{P}}$ to M at the point $\mathscr{P}$.

We now turn to systems theory for application of these geometric ideas.

6. ACTION OF THE FEEDBACK GROUP ON LINEAR SYSTEMS

Let us look at the feedback equivalence problem for nonlinear input systems, say of the following form:

$$\frac{dx^i}{dt} = f^i(x,u)\,dt, \quad u = (u^a), \tag{6.1}$$

$$1 \leqslant i,j \leqslant n, \quad 1 \leqslant a,b \leqslant m.$$

Let Z be the space of variables (x^i,u^a,t), i.e.,

$$Z = R^n \times R^m \times R. \tag{6.2}$$

Consider the Pfaffian system on Z generated by the forms

$$\theta^i = dx^i - f^i\,dt. \tag{6.3}$$

Denote such systems by σ. σ is then a free submodule of $\mathscr{D}^1(Z)$ of rank n, i.e., a *Pfaffian system*. Denote it by

$$\mathscr{P}(\sigma).$$

Let Σ denote the space of all systems defined in this way on the manifold Z. Σ is thus parametrized by the n-real-valued, C^∞ functions

$$f^1,\ldots,f^n$$

on $R^n \times R^m$.

We are interested in transforming the forms θ^i given by (6.3), and hence the Pfaffian system they generate, under the feedback group G:

$$x \to \phi(x) = x',$$

$$u \to \eta(x,u) = u'.$$

The Lie algebra $\mathscr{G}$ of this group is the set of vector fields of the following form:

$$V = a^i(x)\frac{\partial}{\partial x^i} + \beta^a(x,u)\frac{\partial}{\partial u^a}, \tag{6.4}$$

where $a^i(x)$ and $\beta^i(x,u)$ are C^∞ functions of the indicated variables.

Let us calculate the Lie derivatives

$$\begin{aligned}\mathscr{L}_V(\theta^i) &= d(\alpha^i) - \left(\alpha^j\frac{\partial f^i}{\partial x^j} + \beta^a\frac{\partial f^i}{\partial u^a}\right)dt \\ &= \frac{\partial \alpha^i}{\partial x^j}dx^j - \left(\alpha^j\frac{\partial f^i}{\partial x^j} + \beta^a\frac{\partial f^i}{\partial u^a}\right)dt \\ &\equiv \left(\frac{\partial \alpha^i}{\partial x^j}f^j - \alpha^j\frac{\partial f^i}{\partial x^j} + \beta^a\frac{\partial f^i}{\partial u^a}\right)dt, \quad \mathrm{mod}\ \mathscr{P}(\sigma).\end{aligned} \tag{6.5}$$

Set

$$W^u = f^i\frac{\partial}{\partial x^i}. \tag{6.6}$$

Thus, W^u is a parametrized vector field in the variables (x^i), with (u) as the parameter. Set

$$A = \alpha^i\frac{\partial}{\partial x^i}, \tag{6.7}$$

a vector field on the space of (x^i);

$$B = \beta^a \frac{\partial}{\partial u^a}, \tag{6.8}$$

a family of vector fields on the space of the (u^a), parametrized by (x). Then, (6.5) can be written in a more coordinate-free way as follows:

$$\mathcal{L}_V(\theta^i) = dx^i([A,W] + \mathcal{L}_B(W))\,dt, \tag{6.9}$$

where $\mathcal{L}_B(W)$ denotes the Lie derivative (in the variables u alone) of the W by the vector field B.

Now, the tangent space Σ_σ to the family of output systems of form (6.1) can be identified with a set

$$(F^1(x,u),\ldots,F^n(x,u))$$

of functions of (x,u). We can then state these geometric developments in the following more concrete way.

Theorem 6.1: Suppose that

$$\sigma: \frac{dx^i}{dt} = f^i(x,u) \tag{6.10}$$

is an input system with state vector

$$x = (x^i) \in R^n$$

and control vector

$$u = (u^i) \in R^m.$$

Let ϵ be a real parameter, $0 \leqslant \epsilon \leqslant 1$, and let

$$\sigma_\epsilon: \frac{dx^i}{dt} = f^i(x,u;\epsilon) \tag{6.11}$$

be a family of systems depending mostly on the parameter ϵ, and reducing to σ at $\epsilon = 0$. The tangent vector to the curve $t \to \sigma_\epsilon$ in Σ is then represented by the functions

$$F^i(x,u) = \frac{\partial f^i}{\partial \epsilon}(x,u,0). \tag{6.12}$$

Suppose that one can find functions $(a^i(x), \beta^a(x,u))$ such that

$$\frac{\partial \alpha^i}{\partial x^j} f^j - \alpha^j \frac{\partial f^i}{\partial x^j} + \beta^a \frac{\partial f}{\partial u^a} = F^i(x,u). \tag{6.13}$$

Then, one can find a one-parameter group

$$t \to g^1(\epsilon) = \exp(\epsilon V)$$

of feedback transformations acting on the space of systems Σ such that the curve

$$t \to \sigma_\epsilon^1 = g^1(-\epsilon)\sigma_\epsilon \tag{6.14}$$

in the space of all systems has tangent vector zero at $\epsilon = 0$.

We can then consider (6.13) as a set of linear equations for the functions (α^i, β^i) of the variables (x,u). We can then iterate the procedure and ask for existence of a one-parameter subgroup

$$\epsilon \to g_1(\epsilon)$$

in G such that

$$\epsilon \to \sigma_\epsilon^2 = g^2(-\epsilon)g^1(-\epsilon)\sigma_\epsilon \tag{6.15}$$

has first- and second-order tangent vectors equal to zero at $\epsilon = 0$. Continuing in this way, we may find "formal" expansions:

$$\sigma_\epsilon = g^1(\epsilon)\, g^2(\epsilon) \cdots (\sigma), \tag{6.16}$$

where σ is a system which is independent of σ. If σ is a *linear* system, and if the formal expansions (6.16) can be made to *converge*, we obtain *linearizations*. Of course, in other works[25,32] linearizations have been found by explicitly constructing the feedback transformations by solving a linear partial differential equation, but the material in this section provides a "geometric" setting for these procedures, and possibly also a *systematic* way of algebracizing the various *ad hoc* procedures for finding asymptotic and/or approximating expansions used in the applied mathematics and mathematical physics literature.

There is another reasonable way of proving the existence of these linearizing maps *rigorously*: The use of one of the functional analysis implicit function theorems.[1,2] This involves choosing the notion of "smoothness" for the maps involved in construction of the infinite-dimensional manifolds in such a way that these techniques can be applied. Working this out in nontrivial situations is a technology of its own, and I will not attempt to enter into it here.

7. CARTAN–VESSIOT FILTRATIONS

We have just seen that the problem of deformation/bifurcation/linearization of nonlinear input systems can be put into a general setting of the theory of geometric structures. I now want to develop an algebraic structure adapted to the study of Pfaffian systems and their deformations, abstracted out of Refs. 17 and 45–47.

Recall first some terminology from linear algebra: Let V be a vector space. An *ascending filtration* on V is a sequence of linear subspaces $V^0, V^1, V^2, \ldots$ of V such that the following conditions are satisfied:

$$\text{(a)} \quad V^0 \subset V^1 \subset V^2 \subset \cdots, \tag{7.1}$$

$$\text{(b)} \quad V = \bigcup_{n=0}^{\infty} V_n. \tag{7.2}$$

Now, we postulate a Lie algebra structure for V; hence change notation $V \to (L)$, and consider a relation between the subspaces of the filtration and the Lie algebra bracket $[\ ,\ ]$.

Definition: Let L be a Lie algebra with bracket $[\ ,\ ]$. An ascending filtration

$$L^0 \subset L^1 \subset L^2 \subset \cdots \tag{7.3}$$

of linear subspaces for L is said to define a *Cartan–Vessiot filtration* if the following conditions are satisfied:

$$[L^j, L^j] \subset L^{j+1} \quad \text{for } j = 0,1,\cdots. \tag{7.4}$$

Note that a *linear subspace* L^0 of a Lie algebra $\mathcal{G}$ canonically defines a Lie algebra L, and an ascending Cartan–Vessiot filtration, in the following way:

$$L = \text{smallest Lie subalgebra of } \mathcal{G} \text{ containing } L^0, \tag{7.5}$$

$$L^1 \equiv L^0 + [L^0, L^0], \tag{7.6}$$

$$L^2 = (L^1)^1 = L^1 + [L^1, L^1], \tag{7.7}$$

and so on.

In differential geometry, this construction has appeared in the work of Goursat, Cartan, and Vessiot[17,45,47] on Pfaffian systems. Let Z be a finite-dimensional, C^∞, paracom-

pact manifold,

$$\mathscr{G} = \mathscr{V}(Z)$$
$$= C^\infty, \text{ real-valued vector fields.}$$

$\mathscr{G}$ is also a $\mathscr{F}(Z)$-module. Suppose

L^0 is an $\mathscr{F}(Z)$-submodule of $\mathscr{V}(Z)$.

Its annihilator

$$\mathscr{P}(L^0) = \{\theta \in \mathscr{D}^1(Z):\ \theta(L^0) = 0\} \tag{7.8}$$

in $\mathscr{D}^1(M)$ defines a *Pfaffian system* on Z. The $L^1 \subset L^2 \subset \cdots$ defined by (7.5)–(7.7) *automatically* are also $\mathscr{F}(Z)$-submodules of $\mathscr{V}(Z)$. They are called the *derived Pfaffian systems* of $\mathscr{V}(Z)$.

8. THE STRUCTURE TENSORS ASSOCIATED WITH CARTAN–VESSIOT FILTRATIONS

Let us return to the general algebraic study of a Cartan–Vessiot filtered Lie algebra:

$$L^0 \subset L^1 \subset L^2 \subset \cdots \subset L, \tag{8.1}$$

$$[L^j, L^j] \subset L^{j+1}, \quad j = 0,1,\cdots. \tag{8.2}$$

Define skew-symmetric bilinear maps

$$\tau_j:\ L^j \wedge L^j \to L^{j+1}/L^j, \quad j = 0,1,\cdots, \tag{8.3}$$

as follows:

$$\tau_j(A_1, A_2) = [A_1, A_2], \quad \text{mod } L^{j+1} \quad \text{for } A_1, A_2 \in L^j. \tag{8.4}$$

The sequence

$$\tau_0, \tau_1, \cdots$$

is called the *bilinear structure tensor* associated with the filtration.

One can also define multilinear structure tensors:

$$\tau^j = L^0 \times \cdots \times L^{j+2} \to L^{j+1}/L^j \tag{8.5}$$

as follows:

$$\tau^0(A_1, A_2) \equiv \tau_0(A_1, A_2)$$
$$\equiv [A_1, A_2], \quad \text{mod} L^0,$$
$$= A_0(A_1)(A_2), \quad \text{mod} L^0, \quad \text{for} A_1, A_2 \in L^0. \tag{8.6}$$

Then

$$\tau^j(A_1, A_2, \ldots, A_{j+2}) = \text{Ad}(A_1)\text{Ad}(A_2)\cdots\text{Ad}(A_{j+2}), \quad \text{mod } L^j, \tag{8.7}$$

$$j = 0,1,2,\cdots, \quad A_1, A_2, \ldots, A_{j+2} \in L^0.$$

As indicated in Ref. 39, these multilinear maps (which, as we shall see in the next section, have a "tensorial" character in the geometrically relevant cases) are the basic invariants for the *equivalence problem*, in the sense of Lie and Cartan.

Remark: If this process is specialized to the Pfaffian system associated with linear input systems, one obtains the *Brunovsky feedback invariants* for linear input systems.[26]

9. CARTAN–VESSIOT FILTRATION WITH A MODULE STRUCTURE

So far, we have been dealing with vector spaces over a field of scalars and with Lie algebra structures defined over these fields, i.e., the following rules are satisfied:

$$[aA_1, A_2] = a[A_1, A_2]$$
$$= [A_1, aA_2], \tag{9.1}$$

for $A_1, A_2 \in L$, $a \in$ scalar field.

In differential geometry, one encounters generalizations of (9.1), where "a" is an element of a ring (possibly even noncommutative), which acts on the Lie algebra L, to make L an $\mathscr{A}$-module. I will now briefly indicate how such a structure can be introduced in a way compatible with the Lie algebra structure and filtrations introduced previously.

I will also briefly indicate how this can be used "geometrically," in terms of vector bundles on manifolds.

Definition: Let L be a Lie algebra, and let $\mathscr{A}$ be a ring, i.e., $\mathscr{A}$ has a multiplication

$$(a_1, a_2) \to a_1 a_2,$$

which is associative, but not necessarily commutative. Suppose that L is a Lie algebra and a (left) $\mathscr{A}$-module, i.e., there is a bilinear map

$$(a, A) \to aA$$

of

$$\mathscr{A} \times L \to L$$

such that

$$a_1(a_2 A) = (a_1 a_2)A \quad \text{for } a_1, a_2 \in \mathscr{A},\ A \in L. \tag{9.2}$$

Then, the $\mathscr{A}$-module and Lie algebra structure on L are *compatible* if the following conditions are satisfied:

For $a \in \mathscr{A}$, $A_1, A_2 \in L$, $[A, aA_2] - a[A_1, A_2]$ lies in the smallest $\mathscr{A}$-submodule of L containing A_1 and A_2. (9.3)

We can utilize these properties in the following way:

Theorem 9.1: Let L be a Lie algebra with an $\mathscr{A}$-module structure compatible with the Lie algebra structure, in the sense above, and let

$$L^0 \subset L^1 \subset \cdots \tag{9.4}$$

be a Cartan–Vessiot filtration of L with the following additional property:

Each $L^j, j = 0,1,\cdots$, is an $\mathscr{A}$-submodule of L, i.e.,

$$aL^j \subset L^j \quad \text{for } a \in \mathscr{A}. \tag{9.5}$$

Then the structure tensor mapping:

$$\tau_j:\ L^j \times L^j \to L^{j+1}$$

is an $\mathscr{A}$-bilinear mapping, i.e.,

$$\tau_j(a_1 A_1, a_2 A_2) = a_1 a_2 \tau_j(A_1, A_2)$$
$$\text{for } a_1, a_2 \in \mathscr{A}, \quad A_1, A_2 \in L_j, \quad j = 0,1,\cdots. \tag{9.6}$$

Proof: This is an obvious consequence of condition (9.3).

Remark: Theorem 9.1 is "trivial," i.e., is an immediate consequence of the "axioms" used to define the object it deals with. However, it has important geometric consequences, proving the *tensorial* nature of various integrability terms encountered in differential geometry. Specialize L and $\mathscr{A}$ as follows:

$\mathscr{A}$ is the commutative associative algebra of C^∞, real-valued functions on a manifold Z.

L is the Lie algebra and $\mathscr{A}$-module of C^∞ vector fields on Z.

$L^0 \subset L^1 \subset \cdots$ is a Cartan–Vessiot filtration, such that each L^j is a locally free submodule of L, i.e., L^j can be identified with the C^∞ cross sections of a nonsingular vector bundle E_j on Z. Then, the structure maps τ_j are generated by bilinear, skew-symmetric bundle maps:

$$E_j \times E_j \to E_j.$$

I will not do so here, but will turn to a sketch of a *deformation theory* of Cartan–Vessiot filtrations.

10. THE DEFORMATION THEORY OF CARTAN–VESSIOT FILTRATIONS OF LIE ALGEBRAS

Deformation theory has been applied successfully to both *geometric* and *algebraic* structures. Having isolated an algebraic structure involved in the Cartan-Vessiot work, it would be interesting to apply algebraic deformation theory to it. I will not attempt that ambitious program here, but will sketch a deformation theory of the Cartan–Vessiot structure (which is a fundamental one for system theory!) along the lines of the standard Lie algebra deformation and equivalence theory. Let L be a Lie algebra, $\mathscr{A}$ a commutative ring. Suppose L has a $\mathscr{A}$-module structure which is compatible with the Lie algebra structure.

In this section we shall deal with Cartan–Vessiot filtrations

$$L^0 \subset L^1 \subset \cdots \subset L, \tag{10.1}$$

satisfying the following condition:

Each L^j is a free $\mathscr{A}$-module of rank n_j. L is a free $\mathscr{A}$-module of rank n, $\quad n_0 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n$. (10.2)

Consider the integers $n_0, n_1, \cdots$ as fixed, and let Γ be the set of Cartan–Vessiot filtrations satisfying these conditions.

Let G be a group, with a representation

$$(g,a) \to ga, \quad (g,A) \to gA$$

by G on $\mathscr{A}$ and L such that for each $g \in G$:

$a \to ga$ is a ring isomorphism,

$A \to gA$ is a Lie algebra isomorphism,

$$g(aA) = g(a)\, g(A). \tag{10.3}$$

Each $g \in G$ then sends such a Cartan–Vessiot filtration $L^0 \subset L^1 \subset \cdots$ into another one

$$gL^0 \subset gL^1 \subset \cdots.$$

We thus have a *transformation* group action of G on Γ. As in other deformation theories, our goal is to parametrize the orbits of G acting on Γ and the orbit space $G \backslash \Gamma$. As we have seen earlier, the first step is to identify the tangent and normal space to each orbit.

Suppose then that

$$\gamma_0 = (L_0^0 \subset L_0^1 \subset \cdots) \tag{10.4}$$

is an element of Γ. Suppose that

$$t \to \alpha(t), \quad 0 \leqslant t \leqslant 1,$$

is a one-parameter family of linear maps. $L \to L$ such that

$$\alpha(t)(aA) = a\alpha(t)(A) \tag{10.5}$$

for $a \in \mathscr{A}$, $\quad A \in L$,

i.e., each $\alpha(t)$ is $\mathscr{A}$-linear,

$$\alpha(0) = \text{identity map}, \tag{10.6}$$

$$[\alpha(t)L_0^j, \alpha(t)L_0^j] \subset \alpha(t)L_0^j \quad \text{for } j = 0,1,2,\cdots, \tag{10.7}$$

or

$$\alpha(t)^{-1}[\alpha(t)L_0^j, \alpha(t)L_0^j] \subset L_0^{j+1} \tag{10.8}$$

for $j = 0,1,2,\cdots$.

Set

$$\beta = \frac{d}{dt}\alpha(t)|_{t=0}. \tag{10.9}$$

Differentiate both sides of (10.8) with respect to t, and set $t = 0$:

$$-\beta([A_1,A_2]) + [\beta(A_1),A_2] + [A_1,\beta(A_2)] \in L_0^{j+1} \tag{10.10}$$

for $A_1, A_2 \in L_0^j$.

For each integer j, let

$$\beta_j\colon\ L_0^j \to L_0^{j+1}/L_0^j \tag{10.11}$$

be the map which results from applying β to L_0^j, then *reducing modulo* L_0^j. We can then apply this to (10.10), obtaining the following result.

Theorem 10.1: The maps β_j indicated in (10.11) (which algebraically are maps of the "filtered" to a "graded" structure) satisfy the following cocyclelike condition:

$$-\beta_{j+1}([A_1,A_2]) + [\beta_j(A_1),A_2] + [A_1,\beta_j(A_2)] = 0 \tag{10.12}$$

for $A_1, A_2 \in L_0^j$, $\quad j = 0,1,2,\cdots$.

Thus, Γ_{γ_0}, the tangent space to the set of Cartan–Vessiot structures with given indices $(n_0, n_1, \cdots)$ can be considered to be the vector space of the maps $(\beta_0, \beta_1, \cdots)$ satisfying (10.12).

We can now describe the tangent space $(G\gamma_0)_{\gamma_0}$ to the orbit $G\gamma_0$ of the group G.

Theorem 10.2: Let the Lie algebra $\mathscr{G}$ of G act on L via Lie derivative

$$\beta(A) = \frac{d}{dt}\exp(tB)(A)|_{t=0} \tag{10.13}$$

for $B \in \mathscr{G}$.

Then the tangent space $(G\gamma_0)_{\gamma_0}$ to the orbit $(G\gamma_0)$ for the group G is the space of such (β_j) such that

$$\beta_j(A) = B(A) \quad \text{for } A \in L_0^j, \tag{10.14}$$

where B is an element (independent of j) of the Lie algebra $\mathscr{G}$.

11. CLOSING REMARKS

Just as in the Kodaira–Spencer theory,[3] we have deduced an algebraic substructure to the problem of deformation of the sort of geometric structures involved in the deformation-equivalence theory of Pfaffian systems. The "first-order obstructions," i.e., the normal vector space to the orbits of the equivalence group, are then computable in algebraic terms. In certain cases, these obstructions can be related to the Gel'fand–Fuks cohomology groups,[11] or computed by generalizations of their technique.

In this first paper, I have indicated how some of the linearization problems of nonlinear control-system theory fit into this general framework.

[1]J. Marsden and N. McCracken, *The Hopf Bifurcation and Its Applications* (Springer-Verlag, New York, 1976).
[2]S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory* (Springer-Verlag, New York, 1982).
[3]K. Kodaira and D. C. Spencer: (a) "On deformations of some complex pseudogroup structures," Ann. Math. **71**, 16–47 (1960); (b) "On deformation of complex structures, I–III," Ann. Math. **67**, 328–466 (1958); **71**, 43–76 (1960); (c) "Deformation of structures on manifolds defined by transitive continuous pseudogroups," Ann. Math. **76**, 306–445 (1962).
[4]E. Cartan, *Les systemes exterieures et leurs applications géométriques*, (Hermann, Paris, 1946).
[5]E. Inonu and E. P. Wigner, "On the contraction of groups and their representations." Proc. Natl. Acad. Sci. USA **39**, 510–24 (1953).
[6]S. Sternberg, "On the structure of local homeomorphism," Am. J. Math. **80**, 623–31 (1958).
[7]K.-T. Chen, "Equivalence and decomposition of vector fields about an elementary critical point," Am. J. Math. **85**, 693–722 (1963).
[8]R. Hermann, "The formal linearization of a semisimple Lie algebra of vector fields about a singular point," Trans. Am. Math. Soc. **130**, 105–9 (1968).
[9]R. Hermann, "Formal linearization of vector fields and related cohomology, I and II," J. Diff. Geom. **8**, 1–14, 15–24 (1973).
[10]V. W. Guillemin and S. Sternberg, "Remarks on a paper of Hermann," Trans. Am. Math. Soc. **130**, 110–6 (1968).
[11]I. M. Gel'fand and D. B. Fuks, "Cohomology of the Lie algebra of tangential vector fields of a smooth manifold," Func. Anal. Appl. **3**, 184–210 (1969).
[12]B. Anderson and J. Moore, *Linear Optimal Control* (Prentice-Hall, Englewood Cliffs, NJ, 1971).
[13]R. Hermann, "On the accessibility problem of control theory," *Proceedings of the Symposium on Differential Equations* (Academic, Colorado Springs, 1961).
[14]R. Hermann, "Some remarks on the geometry of systems," *Proceedings of the NATO Institute on Geometric and Algebraic Methods for Non-Linear Systems, Imperial College*, edited by D. Mayne (D. Reidel, Boston, 1973).
[15]R. Hermann, *Differential Geometry and the Calculus of Variations*, Vol. 17, Interdisciplinary Mathematics (Math. Sci. Press, Brookline, MA, 1977), 2nd ed.
[16]S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, Englewood Cliffs, NJ, 1964).
[17]E. Goursat, *Leçons sur le Probleme de Pfaff* (Hermann, Paris, 1922).
[18]J. L. Sedwick and D. L. Elliott, "Linearization of analytic vector fields in the transitive case," J. Diff. Eq. **25**, 370–90 (1977).
[19]B. Jakubczyk and W. Respondek, "On the linearization of control systems," Bull. Acad. Polon. Sci., Ser. Sci., Math. Astronom. Phys. **28**, 517–22 (1980).
[20]A. Isidori and A. J. Krener, "On the feedback equivalence of nonlinear systems," Systems Control Lett. **2**, 118–21 (1982).
[21]A. J. Krener, "On the equivalence of control systems and the linearization of nonlinear systems," SIAM J. Control **11**, 670–6 (1973).
[22]R. W. Brockett, "Feedback invariants for nonlinear systems," IFAC Congress, Helinski, 1978.
[23]L. R. Hunt and R. Su, "Linear equivalents of nonlinear time varying systems, *International Symposium on the Mathematical Theory of Networks and Systems, Santa Monica, 1981* (Western Periodical Co., North Holywood, CA, 1981) pp. 119–23.
[24]G. Meyer and L. Cicolani, "A formal structure for advanced automatic flight control systems," NASA TN D-7940, 1975.
[25]J. R. Hunt, R. Su, and G. Meyer, "Global transformations of nonlinear systems," IEEE Trans. Auto. Cont. **27**, 6 (1982).
[26]P. Brunovsky, "A classification of linear controllable systems," Kibern. (Praha) **6**, 173–88 (1970).
[27]R. E. Kalman, *Kronecker Invariants and Feedback, Ordinary Differential Equations*, edited by L. Weiss (Academic, New York, 1972), pp. 459–71.
[28]G. Meyer and L. Cicolani, "Applications of nonlinear system inverses to automatic flight control design—system concepts and flight evaluations," *AGARDograph 251 on Theory and Applications of Optimal Control in Aerospace Systems*, edited by P. Kant (NATO, Paris, 1980).
[29]G. Meyer, "The design of exact nonlinear model followers," Proceedings 1981 Joint Automatic Control Conference FA3A.
[30]G. A. Smith and G. Meyer, "Total aircraft flight control system balanced open- and closed-loop with dynamic trimmaps," 3rd Avionics Conference, 1979.
[31]G. A. Smith and G. Meyer, "Applications of the concept of dynamic trim control to automatic landing of carrier aircraft," NASA TP-1512, 1980.
[32]R. Su, "On the linear equivalents of nonlinear systems," Systems Control Lett. **2** (1982).
[33]W. R. Wehrend, Jr. and G. Meyer, "Flight tests of the total automatic flight control system (TAFCOS) concept on a DHC-6 Twin Otter aircraft," NASA TP-1513, 1980.
[34]W. R. Wehrend, Jr., "Pilot control through the TAFCOS automatic flight control system," NASA TM-81152, 1979.
[35]W. M. Wohham and A. S. Morse, "Feedback invariants of linear multivariable systems," Automatica **8**, 93–100 (1972).
[36]W. Respondek, "Geometric methods in linearization of control systems" (to be published).
[37]E. S. Livingston and D. L. Elliott, "Linearization of families of vector fields" (to be published).
[38]A. J. Krener and A. Isadori, "Linearization by output injection and nonlinear observers" (to be published).
[39]R. Hermann, "The theory of equivalence of Pfaffian systems and input systems under feedback," Math. Systems Theory **15**, 343–56 (1982).
[40]A. Nijenhuis and R. Richardson, "Deformations of homomorphisms of Lie groups and Lie algebras," Bull. Am. Math. Soc. **73**, 175–9 (1967).
[41]R. Richardson, "Deformations of Subalgebras of Lie algebras," J. Diff. Geom. **3**, 298–308 (1969).
[42]V. Guillemin and S. Sternberg, *Deformation Theory and Pseudogroup Structures*, Memoirs of the Am. Math. Soc. (AMS, Providence, RI, 1965).
[43]R. Hermann, "Analytic continuation of group representation, I–VI," Comm. Math. Phys. **2**, 251–70 (1966); **3**, 53–74, 75–97 (1966); **5**, 131–56, 157–90 (1967); **6**, 205–25 (1967).
[44]C. Ehresmann: (a) "Sur les structures infinitésimales régulières," Cong. Int. Math. Amsterdam **1**, 479–80 (1954); (b) "Connexions infinitésimales," Colloq. Top. Alg. Bruxelles, 29–55 (1950); (c) "Structures infinitésimales et pseudogroupes de Lie," Coll. Int. C. N. R. S. Geom. Diff. Strasbourg, 97–110 (1953); (d) C. R. Acad. Sci. Paris **240**, 1762 (1954); **241**, 397, 1755; (1955); **246**, 360 (1958). (e) "Connexions d'ordre supérieur," *Atti 5 Congreso. dell' Unione Matematica Italiana* 1955 (Cremonese, Rome, 1956), pp. 326–8; (f) "Catégories topologiques et catégories différentiables," Colloq. Géom. Diff. Globale Bruxelles, C. B. R. M., 137–50 (1958); (g) "Groupoides differenciales," Rev. Un. Mat. Argentina, Buenos-Aires **19**, 48 (1960).
[45]E. Cartan, *Oeuvres Completes, Part II* (Gauthier-Villars, Paris, 1955).
[46]R. Hermann, "Cartan connections and the equivalence problem," J. Diff. Eq. **3**, 199–248 (1964).
[47]E. Vessiot, "Sur une théorie nouvelle des problèmes généraux d'integration," Bull. Soc. Math. Fr. **52**, 336–96 (1924).

The isospectral property for a family of non-self-adjoint operators

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We prove the isospectral property for certain families of linear non-self-adjoint operators which play a role in inverse scattering theory for a class of nonlinear evolution equations of interest in physics. These include the sine-Gordon and nonlinear Schrödinger equations.

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The isospectral problem plays a central role in the application of inverse scattering theory to nonlinear evolution equations (see, for instance, Ref. 1) and originates in the pioneering discovery of Gardner *et al.*[2] that the spectrum of the linear operator $L(t)$ associated to the (nonlinear) Korteweg–de Vries (KdV) equation is independent of time t. In this case, $L(t)$ is a family (parametrized by t) of *self-adjoint* operators of Schrödinger type, and this invariance—referred to as the "isospectral property" of the (nonlinear) flow—is proved through construction of a family $U(t)$ of unitary operators such that

$$U(t)L(t)U(t)^{-1} = L(0). \tag{1}$$

The above-mentioned construction proceeds as follows. One proves that there exist operators $M(t)$ such that, for the moment, formally,

$$\frac{dL(t)}{dt} = [M(t), L(t)]. \tag{2}$$

This is sometimes abbreviated by saying that $\{L, M\}$ form a "Lax pair," because Lax proved in his fundamental paper[3] that (2) is in fact equivalent to the KdV equation (for the corresponding choices of L and M).

From (2), M is seen to be necessarily (formally) skew-adjoint: in fact, (1) and (2) are equivalent, with

$$M(t) = -U(t)^{-1}\frac{dU(t)}{dt}. \tag{3}$$

For a large class of evolution equations, however, L is not self-adjoint. This occurs with the generalization by Ablowitz *et al.*[4] of the Zakharov–Shabat[5] systems, which include the sine-Gordon and nonlinear Schrödinger equations. In Ref. 4 the isospectral property for a class of $L(t)$ was *assumed* and the consistency equations derived under this assumption were shown to be equivalent to a large class of evolution equations. This is consistent with, but does not *prove*, the isospectral property for the operators involved, because at least one component of the pair (usually M) depends parametrically on the eigenvalue. Fortunately, there exist "true" Lax pairs for many evolution equations considered in Ref. 4, that is, with both $L(t)$ and $M(t)$ independent of the eigenvalue. For the sine-Gordon equation they were constructed in Ref. 6, and for the nonlinear Schrödinger equation, in Ref. 5. Such treatments seem, at the moment, indispensable for a rigorous mathematical treatment of the application of inverse scattering theory to these equations. In these cases, however, $L(t)$ remains not self-adjoint, and $M(t)$ not skew-adjoint, so that the isospectral property must be proven differently. We consider here a theorem on the invariance of the discrete spectrum, which is enough for applications. Invariance of the whole spectrum follows from general properties of the family $L(t)$: this problem will be studied elsewhere.[7]

Finally, we remark that there exists, as yet, no general method to find Lax pairs: see, however, Ref. 8, where a method is developed for a class of evolution equations. It is, however, an open problem to find such pairs for several nonlinear equations of physical relevance, such as the Maxwell–Bloch equations of self-induced transparency,[9] but we believe the structure described below [Eqs. (4)–(6)] to be applicable to a wide class of nonlinear equations.

We conclude this introduction with two remarks of general nature concerning Lax pairs. First, it has recently been proven that for Hamiltonian systems, the dynamical equations may always be written in the special Lax form[10] (2). Second, Eq. (2) is of great importance in the general theory of completely integrable systems, allowing a unified derivation of conservation laws (see, e.g., Ref. 11 for a nice review). Using our results, the latter method yields a rigorous derivation of conservation laws for the sine-Gordon and nonlinear Schrödinger equations.

In this paper, we shall deal with "small perturbations of self-adjointness (and skew-adjointness)" in the following sense. We shall be working in a fixed Banach space X, and denote by $D(A)$ and $\sigma_d(A)$ the domain and the discrete spectrum (i.e., the set of isolated eigenvalues of finite multiplicity) of an operator A on X. The operators $L(t)$ and $M(t)$ forming a Lax pair will be assumed throughout to have the form

$$L(t) = L_1 + L_2(t), \tag{4}$$

$$M(t) = M_1 + M_2(t), \tag{5}$$

where L_1 is self-adjoint on $D(L_1)$, M_1 is skew-adjoint on $D(M_1)$, and $L_2(t)$ and $M_2(t)$, are, for each $t \in \mathbb{R}$, *bounded* operators on X. We further assume that

$$D(M_1L_1) \subseteq D(M_1^2) \cap D(M_1). \tag{6}$$

To simplify proofs, and because it holds in the examples

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we shall consider, we finally assume that

$$D(M_1)\subseteq D(L_1). \tag{7}$$

[Hence $D(L_1M_1)=D(L_1(M_1+\zeta)^{-1}(M_1+\zeta)M_1)\supseteq D(M_1^2)\cap D(M_1)$ by (6), where $\zeta\in\mathbb{R}$, $\zeta\neq 0$.] Proofs may, however, be modified if (7) does not hold, yielding similar results.

We now describe briefly the strategy of the proof, which is very simple. Let $\xi_0\in\sigma_d(L(0))$. Under certain assumptions, the isolated eigenvalues of $L(0)$ are simple.[7] Let u_0 be the associated eigenvector, and suppose

$$u_0\in D(M_1). \tag{8}$$

For each $t\in\mathbb{R}$, define the vector

$$f(t)\equiv L(t)u_0(t)-\xi_0u_0(t), \tag{9}$$

where $u_0(t)$ is the (hopefully) unique solution of the initial-value problem

$$\frac{du(t)}{dt}=M(t)u(t),\quad u(0)=u_0\in D(M_1). \tag{10}$$

Then by (2), we have, formally,

$$\frac{df(t)}{dt}=M(t)f(t),\quad f(0)=0. \tag{11}$$

By uniqueness, $f(t)=0\ \forall\ t\in\mathbb{R}$, and hence $\sigma_d(L(0))\subseteq\sigma_d(L(t))$.

Reversing the arguments, $\sigma_d(L(t))\subseteq\sigma_d(L(0))$.

In order to render the above formal arguments rigorous, some additional assumptions are required. In order to control the problem of existence and uniqueness of the initial-value problem (10), we use theorem X-70 of Ref. 12, which is reproduced for convenience as Theorem A1 of the Appendix.

Assumption (a) of Theorem A1 is satisfied if a suitable real constant ζ is added to M_1, which does not alter the results (see Ref. 12, p. 286). This constant might depend on t (because we are not assuming $\|M_2(t)\|\leqslant c<\infty$ for c independent of $t\in\mathbb{R}$), but it may be chosen uniformly in t for t in any compact subinterval of R, by the forthcoming assumption (B), and this suffices (see Theorem 1). We denote by $B(X)$ the set of bounded operators on X and define the formal operator families:

$$F(t)\equiv[L_1,M_2(t)](M_1+\zeta)^{-1}, \tag{12a}$$

$$G(t)\equiv[M_1,L_2(t)](M_1+\zeta)^{-1}, \tag{12b}$$

$$H(t)\equiv[M_1,M_2(t)](M_1+\zeta)^{-1}, \tag{12c}$$

where $[A,B]$ denotes the commutator of two operators A and B on X, and $\zeta\in R$ is such that $(M(t)+\zeta)$ satisfies (a) of Theorem A1 and may vary with the compact t-subinterval as discussed previously. We denote by $B(X)$ the set of bounded operators on X and state for convenience:

Definition 1: An operator family $\{A(t)\in B(X);t\in R\}$ is said to be smooth if $A(t)$ is strongly continuously differentiable in $t\in\mathbb{R}$ and uniformly bounded in t in the operator norm for t in compact subintervals of $\mathbb{R}$.

Our main additional assumptions may now be stated:

(A) $L_2(t)$,

(B) $M_2(t)$,

(C) $F(t)$,

(D) $G(t)$, and

(E) $H(t)$

are smooth operator families.

By (5) and assumption (B), it follows that assumptions (b) and (c) of Theorem A1 are satisfied for the initial-value problem (10) which has, therefore, a unique solution which we denote by $u_0(t)$.

Proposition 1: If $u_0\in D(M_1^2)\cap D(M_1)$ in (10), then $L_1u_0(t)$ is continuously differentiable in t for t in any open subinterval of $\mathbb{R}$.

Proof: Let $x(t)\equiv(M_1+\zeta)u_0(t)$. *Formally* $x(t)$ would satisfy the differential equation

$$\frac{dx(t)}{dt}=(M_1+\zeta)M(t)(M_1+\zeta)^{-1}x(t) \tag{13a}$$

with initial value

$$x(0)=(M_1+\zeta)u_0\in D(M_1). \tag{13b}$$

By (6) and assumption (E), the operator family

$$R(t)\equiv(M_1+\zeta)M(t)(M_1+\zeta)^{-1}=(M_1+\zeta)+S(t),$$

where

$$S(t)=H(t)+M_2(t)$$

satisfies assumption (a) of Theorem A1 (except for eventually modifying ζ as was discussed before). We have, in the notation of the Appendix,

$$\begin{aligned}C(t,s)&=(1/(t-s))(R(t)-R(s))R(s)^{-1}\\&=(1/(t-s))(S(t)-S(s))R(s)^{-1}.\end{aligned}$$

It is clear that (b) and (c) of theorem A1 follow from assumptions (B) and (E) if we prove that $R(s)^{-1}$ is (1) strongly continuous in s and (2) uniformly bounded for s in compact subintervals of $\mathbb{R}$. Again, we may choose ζ such that $\|S(s)(M_1+\zeta)^{-1}\|\leqslant c<1$ with c chosen uniformly in s in compact subintervals. Hence

$$R(s)^{-1}=(M_1+\zeta)^{-1}(1+S(s)(M_1+\zeta)^{-1})^{-1},$$

and both assertions (1) and (2) follow from the expansion of $(1+S(s)(M_1+\zeta)^{-1})^{-1}$ in a (uniformly in s for s in compact subintervals) norm-convergent power series.

Hence by theorem A1, there exists a unique solution $x_0(t)$ of the initial-value problem (13). Let $\tilde{u}_0(t)\equiv(M_1+\zeta)^{-1}x_0(t)$. By (13) $\tilde{u}_0$ is continuously differentiable and satisfies (10) with initial value u_0 given by (13b). By uniqueness of the solution of (10), $\tilde{u}_0(t)=u_0(t)$. Hence $M_1u_0(t)$ is continuously differentiable, and the same follows for $L_1u_0(t)=L_1(M_1+\zeta)^{-1}(M_1+\zeta)u_0(t)$ by (7). ■

Theorem 1: Suppose that for all $t\in\mathbb{R}$ and $v\in D(M_1^2)\cap D(M_1)$, $L(t)v$ is continuously differentiable, $[M(t),L(t)]v$ is continuous, and the following "Lax pair equation" holds:

$$\frac{dL(t)}{dt}v=[M(t),L(t)]v. \tag{14}$$

Assume further that if u_0 is an eigenvector of $L(0)$ corre-

sponding to a (simple) eigenvalue ξ_0, then

$$u_0 \in D(M_1^2) \cap D(M_1). \tag{15}$$

[Notice that the latter condition is stronger than (8)]. Then

$$\sigma_d(L(t)) = \sigma_d(L(0)). \tag{16}$$

Remark 1: Notice that by (6), (7), and assumptions (A), (B), (C), and (D),

$$[M(t),L(t)]v = [M_1,L_1]v + [M_2(t),L_1]v + [M_1,L_2(t)]v + [M_2(t),L_2(t)]v$$

is well defined and is continuous in t for all $v \in D(M_1^2) \cap D(M_1)$. ■

Proof: Let $I_1 = (-T,T)$, $T < \infty$. We prove that $\sigma_d(L(t)) = \sigma_d(L(0))$ for all $t \in I_1$. By the same method, we prove that $\sigma_d(L(t)) = \sigma_d(L(0))$ for all $t \in I_2 = (T/2, 3T/2)$ and all $t \in I_3 = (-3T/2, T/2)$, and so on, until we arrive at (16).

Set $B = L_1 + i\alpha$, $\alpha \in R$, $\alpha \neq 0$ in Lemma A1 of the Appendix. By (7), (15), and Proposition 1, $Bu_0(t)$ and $Bu_0'(t)$ are defined and continuous, and B is, by construction, defined on $D(L_1)$ with a bounded inverse. Hence by (A1) of Lemma A1, $L(t)u_0(t)$ is continuously differentiable with

$$\frac{d}{dt}(L(t)u_0(t)) = \frac{dL(t)}{dt}u_0(t) + L(t)\frac{du_0}{dt}. \tag{17}$$

By (14) and (17), $f(t)$ defined by (9) is indeed continuously differentiable and (11) holds. By the uniqueness part of Theorem A1, $f(t) = 0$ $\forall$ $t \in I$, and hence $\sigma_d(L(0) \subseteq \sigma_d(L(t))$ $\forall$ $t \in I$. Upon considering the proble, with initial value $t_0 = t \in I$,[13] we obtain $\sigma_d(L(t)) \subseteq \sigma_d(L(0))$, hence $\sigma_d(L(t)) = \sigma_d(L(0))$ $\forall$ $t \in I$. ■

Remark 2: When $[M_1,M_2(t)]$ is a bounded operator with norm $\|[M_1,M_2(t)]\| \leqslant c < \infty$ with c independent of t, the problem of existence and uniqueness of solutions of the initial-value problem (10) may be treated by "interaction representation methods" (see Ref. 12, p. 283). These conditions are not met, however, in the applications we shall consider.

We now present some examples.

(a) Sine-Gordon equation[6]:

$$x = L^2(\mathbb{R}) \otimes \mathbb{C}^4,$$

$$L_1 = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\frac{d}{dx}, \tag{18a}$$

$$L_2(t) = \begin{pmatrix} A(t) & B(t) \\ B(t) & 0 \end{pmatrix}, \tag{18b}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A(t) = \frac{i}{4}\begin{pmatrix} 0 & W(x,t) \\ W(x,t) & 0 \end{pmatrix},$$

$$B(t) = \frac{1}{4}\begin{pmatrix} \exp(iu(x,t)/2) & 0 \\ 0 & \exp(-iu(x,t)/2) \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{d}{dx}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{19a}$$

$$M_2(t) = \begin{pmatrix} 0 & C(t) \\ D(t) & 0 \end{pmatrix}, \tag{19b}$$

where

$$C(t) = 2JB(t), \quad D = -2B(t)J, \tag{20}$$

$$w(x,t) = \frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}, \tag{21}$$

and u is the real-valued infinitely differentiable global solution of the sine-Gordon equation (see Ref. 14, p. 25)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0, \quad -\infty < x,t < \infty \tag{22}$$

with the initial value

$$u(x,o) = f(x) \in C_0^\infty, \tag{23a}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \in C_0^\infty, \tag{23b}$$

where C_0^∞ denotes the Schwartz space of infinitely differentiable functions of compact support. We see that

$$D(L_1) = D(M_1) = D(L(t)) = D(M(t)) = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}; u_i \cdots u_i \text{ absolutely continuous with } \frac{du_i}{dx} \in L^2(R) \text{ for all } i = 1,2,3,4. \right\}$$

Proposition 2: $\sigma_d(L(t)) = \sigma_d(L(0))$ for the sine-Gordon equation in the above formulation.

Proof: We must verify (4), (5), (6), (7), assumptions (A)–(E), (14), and (15). It is completely straightforward to verify (4), (5), (6), (7), and (14). To prove (15), notice that by the eigenvalue equation

$$L(0)u_0 = \xi_0 u_0,$$

it follows that $u_{0i} \in C^\infty(R)$, with $d^n u_{0i}/dx^n \in L^2(R)$ for all integer n and all $i = 1,2,3,4$. This implies (15) because of the explicit form (19a).

To show what is involved in the proof of assumptions (A)–(E), we compute

$$[M_1,M_2(t)] = \begin{pmatrix} 0 & \partial C/\partial x \\ -\partial D/\partial x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2C \\ -2D & 0 \end{pmatrix}\frac{d}{dx}.$$

From the above form, $[M_1M_2(t)]$ is $(M_1 + \zeta)$-bounded,[15] with bounds depending on the L^∞-norms[16] of $(\partial u(x,t)/\partial x)e^{\pm iu(x,t)}$ and $e^{\pm iu(x,t)}$. These functions of C^∞ in x and t under assumption (23) (Ref. 14, p. 45) and are continuously differentiable in t and uniformly bounded in t for t in compact subintervals of R in suitable Sobolev spaces (Ref. 14, pp. 42–45). By the Sobolev embedding theorems (Ref. 14, p. 21), these same properties hold in the sense of the L^∞-norm. This implies that assumption (E)[17] and the proofs of (A)–(D) are similar. ■

(b) Nonlinear Schrödinger equation[5]:

$$X = L^2(\mathbb{R}) \otimes \mathbb{C}^2,$$

$$L_1 = i \begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix} \frac{d}{dx},$$

$$L_2(t) = \begin{pmatrix} 0 & \bar{u}(x,t) \\ u(x,t) & 0 \end{pmatrix},$$

where the bar denotes complex conjugate.

$$M_1 = ip \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d^2}{dx^2},$$

$$M_2(t) = -i \begin{pmatrix} \bar{u}u/(1+p) & i\dfrac{\partial u}{\partial x} \\ -i\dfrac{\partial u}{\partial x} & -\bar{u}u/(1-p) \end{pmatrix},$$

where $p^2 > 0$ and u is the infinitely differentiable global solution of the nonlinear Schrödinger equation (see Refs. 18 and 19).

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{2}{1-p^2}(\bar{u}u)u = 0$$

with initial value

$$u(0,x) = h(x) \in C_0^\infty.$$

We note that $D(L_1) = D(L(t)) = \{\binom{u_1}{u_2}; u_1, u_2$ absolutely continuous; $u_1', u_2' \in L^2(R)\}$ and $D(M_1) = D(M(t)) = \{\binom{u_1}{u_2}; u_1', u_2'$ absolutely continuous; $u_1'', u_2'' \in L^2(\mathbb{R})\}$. Just as in Proposition 2 (but using the results of Ref. 15 and 16), we may verify (4), (5), (6), (7), (14), (15), and assumptions (A)–(E) for the above system and obtain the following.

Proposition 3: $\sigma_d(L(t)) = \sigma_d(L(0))$ for the nonlinear Schrödinger equation in the above formulation.

APPENDIX

In this Appendix, we reproduce some definitions and theorems used in the main text. As in the main text we shall be working in a fixed Banach space X.

We shall denote by $\rho(A)$ the resolvent set of an operator A on X. For each $t \in R$, let $A(t)$ generate a contraction semigroup (Ref. 12, p. 235) on X. For each positive integer k, let the approximate "evolution operators" be defined by

$$U_k(t,s) \equiv \exp(-(t-s)A((i-1)/k)) \quad \text{if}$$
$$(i-1)/k \leqslant s \leqslant t \leqslant i/k, \quad (1 \leqslant i \leqslant k)$$

and

$$U_k(t,r) = U_k(t,s)U_k(s,r) \quad \text{if} \quad 0 \leqslant r \leqslant s \leqslant t \leqslant 1.$$

Theorem A1: (Theorem X-70 of Ref. 12). Let I be an open interval in R. For each $t \in I$ let $A(t)$ be the generator of a contraction semigroup on X so that $0 \in \rho(A(t))$ and (a) the $A(t)$ have a common domain D.

By the closed graph theorem, $A(t)A(s)^{-1}$ is bounded and we define

$$C(t,s) = A(t)A(s)^{-1} - 1.$$

(b) For each $\varphi \in X$, $(t-s)^{-1}C(t,s)\varphi$ is uniformly strongly continuous and uniformly bounded in s and t for $t \neq s$ lying in any fixed compact subinterval of I.

(c) For each $\varphi \in X$, $C(t)\varphi \equiv \lim_{s \uparrow t}(t-s)^{-1}C(t,s)\varphi$ exists uniformly for t in each compact subinterval and $C(t)$ is bounded and strongly continuous in t.

Then for all $s \leqslant t$ in any compact subinterval of I and any $\varphi \in X$,

$$U(t,s)\varphi = \lim_{k \to \infty} U_k(t,s)\varphi$$

exists uniformly in s and t. Further, if $\psi \in D$, then $\varphi_s(t) \equiv U(t,s)\psi \in D$ for all t and satisfies

$$\frac{d\varphi_s(t)}{dt} = -A(t)\varphi_s(t), \quad \varphi_s(s) = \psi,$$

and $\|\varphi_s(t)\| \leqslant \|\psi\|$ for all $t \geqslant s$. ■

Uniqueness is implied by the last inequality.

We also used the following Lemma. For a proof, see, e.g., Lemma 1.3, p. 178 of Ref. 20. The word "function" is understood to mean X-valued function.

Lemma A1: Let the closed operator $A(t)$ have constant domain $D(A)$ and be strongly continuously differentiable on it. Suppose in addition that the closed operator B is defined on $D(A)$ and has a bounded inverse. Suppose finally that the function $f(t)$ is continuously differentiable, and that the functions $Bf(t)$ and $Bf'(t)$ are defined and continuous. Then the function $A(t)f(t)$ is continuously differentiable and

$$\frac{d}{dt}(A(t)f(t)) = \frac{dA(t)}{dt}f(t) + A(t)\frac{df(t)}{dt}. \qquad \text{(A1)}\blacksquare$$

Definition A1: Let A and B be densely defined linear operators on a Banach space X. Then B is said to be A-bounded if

(i) $D(B) \supseteq D(A)$ and
(ii) For some $a,b \in \mathbb{R}$, and all $\varphi \in D(A)$,

$$\|B\varphi\| \leqslant a\|A\varphi\| + b\|\varphi\|.$$

We call a and b the "bounds of B with respect to A," although this is not standard terminology.

If $A^{-1} \in B(X)$ and $\|A^{-1}\| \leqslant \alpha$, it follows from (ii) that

$$\|BA^{-1}\| \leqslant a + b\alpha.$$

In the text, a and b depend on L^∞-norms of some functions h, and we have, therefore, inequalities of the form

$$\|B(t)A^{-1}\| \leqslant \|h(t)\|_\infty.$$

Hence continuous differentiability of $\|B(t)A^{-1}\|$ [which is stronger than the strong continuous differentiability required in (A)–(E)] is implied by continuous differentiability of h in L^∞.

[1] A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE **61**, 1443 (1973).
[2] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[3] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).
[4] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974).
[5] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
[6] L. A. Takhtadzhyan and L. D. Faddeev, Theor. Math. Phys. (USSR) **21**, 160 (1974).

[7]H. M. A. de Castro and W. F. Wreszinski, J. Math. Phys. **24**, 1502 (1983).
[8]N. K. Ibragimov and A. B. Shabat, Functional Anal. Appl. **14**, 25 (1980); **14**, 79 (1980).
[9]G. L. Lamb, Jr., *Elements of Soliton Theory* (Wiley, New York, 1980) or, for particularly clear account, H. M. Nussenzveig, *Introduction to Quantum Opticas* (Gordon and Breach, New York, 1973).
[10]F. A. Berezin and A. M. Perelomov, Functional Anal. Appl. **14**, 50 (1980).
[11]M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71**, 313 (1980).
[12]M. Reed and B. Simon, *Methods of Modern Mathematical Physics Fourier Analysis, Self-Adjointness* (Academic, New York, 1975), Vol. II.
[13]We are using implicitly the fact that the isolated eigenvalues of $L(t)$ are simple for all t. This follows from Lemma 4 of Ref. 7.
[14]M. Reed, *Abstract Nonlinear Wave Equations, Lecture Notes in Mathematics* **507** (Springer, Berlin, 1976).
[15]See Definition A1 of the Appendix.
[16]$\|f\|_\infty \equiv \text{ess sup}_{x\in\mathbb{R}}|f(x)| = \sup_{x\in\mathbb{R}}|f(x)|$ in our case because the f's are continuous.
[17]See also the discussion after Definition A1 in the Appendix.
[18]W. A. Strauss, in *Lecture Notes in Physics* **98** (Springer, Berlin, 1979), p. 123.
[19]J. Ginibre and G. Velo, in *Lecture Notes in Physics* **120** (Springer, Berlin, 1980), p. 218.
[20]S. G. Krein, *Linear Differential Equations in Banach Space*, Amer. Math. Soc. Trans. (Am. Math. Soc., Providence, RI, 1971), Vol. 29.

A new class of integrable systems[a)]

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We present a family of dynamical systems associated with the motion of a particle in two space dimensions. These systems possess a second integral of motion quadratic in velocities (apart from the Hamiltonian) and are thus completely integrable. They were found through the derivation and subsequent resolution of the integrability condition in the form of a partial differential equation (PDE) for the potential. A most important point is that the same PDE was derived through considerations on the analytic structure of the singularities of the solutions ("weak-Painlevé property").

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I. INTRODUCTION

The aim of this paper is to present a "hidden treasure" class of integrable systems in two space dimensions. Given the extreme rarity of integrable dynamical systems, which has spurred an intense research in this domain, and the fact that at least one example of integrable system belonging to the aforementioned class was known since the beginning of celestial mechanics, it is astonishing that these systems have lain undiscovered for over 80 years.

In a 1901 paper, Darboux,[1] applying a method due to Bertrand,[2] obtained the general partial differential (PDE) which the potential must satisfy in order for the system to possess an integral of motion quadratic in the velocities. He then proceeded to specify and solve the equation in the general case, while being aware of the possibility of the existence of a particular case of integrability, the importance of which he did not grasp at the moment. The incomplete former case of the Darboux integrability was subsequently presented by Whittaker[3] as "the only case of the motion of a particle in a plane under the action of conservative forces which possess an integral quadratic in velocities other than the integral of energy." However, the recent discovery by Greene[4] of an integral of motion quadratic in the velocities for the Henon–Heiles system, which can easily be shown not to belong to the Darboux family, weakened the generality of the Darboux solution. Actually, the most classical example of a non-Darboux integral of motion is the third integral of motion for the Kepler problem, which has been known for several centuries.

It must be stressed at this point that integrability in our case of two-dimensional systems means the existence of a second integral of motion besides the energy of the system. However, we explicitly demand that this second integral exist for every value of the energy. An interesting generalization of the concept of integrability was obtained by Hall,[5] by releasing this constraint. In his approach, one can also obtain integrability for a larger class of systems, but only for some specific values of the energy.

A first result of our work was presented elsewhere.[6] It concerned a two-dimensional quintic homogeneous polynomial potential which can be shown to lead to an integrable dynamical system. The particularity of this potential was that the associated equations of motion did not possess the Painlevé property, which has been conjectured, and amply verified, by Ablowitz, Ramani, and Segur,[7] to be associated with integrability. This entailed a weakening of the Painlevé criterion for two-dimensional systems. We then proposed to replace it by a "weak Painlevé property," which is not just an artifice which allows us to explain the particularities of the quintic potential; it has a real predictive power. In the case at hand this property, suitable applied, allows us to derive a PDE to be obeyed by the potential (Sec. III). This equation is precisely the particular case of the PDE discovered by Darboux that the latter did not investigate (Sec. II). In Sec. IV we present the general solution of this PDE, and we display some cases of particular interest.

II. INTEGRALS OF MOTION QUADRATIC IN VELOCITIES

Let us consider the motion of a particle of unit mass in a two-dimensional potential $V(x,y)$. The Hamiltonian governing the system reads

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x,y). \tag{1}$$

The equations of motion associated with the system are simply

$$\ddot{x} = -\frac{\partial V}{\partial x} \equiv -V_x, \quad \ddot{y} = -\frac{\partial V}{\partial y} \equiv -V_y. \tag{2}$$

For the complete integrability of the system, one needs the existence of a second constant of motion, besides the Hamiltonian itself. In this paper we will focus on integrals of motion quadratic in the velocities, although integrals of order higher than two are not unknown.[6,8,9] The general form of such an integral is

$$C = g^{(0)}\dot{x}^2 + g^{(1)}\dot{x}\dot{y} + g^{(2)}\dot{y}^2 + h, \tag{3}$$

[a)] The authors dedicate this work to their Soviet colleague, the mathematician and physicist Nahum Meiman.

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where $g^{(i)}$ and h are functions of x and y. No term linear in velocities is allowed as the Hamiltonian (1) is invariant under time reversal. The conditions of the constancy of C can be written as

$$0\equiv\frac{dC}{dt}=g_x^{(0)}\dot{x}^3+g_y^{(0)}\dot{x}^2\dot{y}+g_x^{(1)}\dot{x}^2\dot{y}+g_y^{(1)}\dot{x}\dot{y}^2+g_x^{(2)}\dot{x}\dot{y}^2 + g_y^{(2)}\dot{y}^3+2g^{(0)}\ddot{x}\dot{x}+g^{(1)}\ddot{y}\dot{x}+g^{(1)}\dot{y}\ddot{x}+2g^{(2)}\ddot{y}\dot{y} + h_x\dot{x}+h_y\dot{y}. \tag{4}$$

Regrouping and equating to zero the coefficients of each monomial in the velocities, we obtain at order three

$$g_x^{(0)}=0,\quad g_y^{(0)}+g_x^{(1)}=0,$$
$$g_y^{(1)}+g_x^{(2)}=0,\quad g_y^{(2)}=0. \tag{5}$$

The solution of this system of equations is straightforward:

$$g^{(0)}=\alpha y^2+\beta y+\gamma,$$
$$g^{(1)}=-2\alpha xy-\beta x-\delta y-\epsilon, \tag{6}$$
$$g^{(2)}=\alpha x^2+\delta x+\zeta.$$

At first order we obtain

$$h_x+2g^{(0)}\ddot{x}+g^{(1)}\ddot{y}=0,$$
$$h_y+g^{(1)}\ddot{x}+2g^{(2)}\ddot{y}=0. \tag{7}$$

The integrability condition for h reads thus:

$$\frac{\partial}{\partial y}[2g^{(0)}V_x+g^{(1)}V_y]=\frac{\partial}{\partial x}[g^{(1)}V_x+2g^{(2)}V_y], \tag{8}$$

or, equivalently,

$$2(g^{(0)}-g^{(2)})V_{xy}+(2g_y^{(0)}-g_x^{(1)})V_x - (2g_x^{(2)}-g_y^{(1)})V_y-g^{(1)}(V_{xx}-V_{yy})=0. \tag{9}$$

This is the equation found by Darboux,[1] which must be obeyed by the potential for the system to be integrable.

The case analyzed by Darboux is obtained by using the general solution (6) for the $g^{(i)}$'s, with the assumption $\alpha\neq 0$. In this case one can, through the adequate translations in the x and y directions, eliminate all the linear terms in (6). Furthermore, we observe that γ and ζ in $g^{(0)}$ and $g^{(2)}$ appear only through $\gamma-\zeta$ in (9). Thus one can take $\zeta=0$ without loss of generality. This is by no means unimportant because $\zeta=0$ ensures that no term of the form $\eta\dot{y}^2$ with η constant will be present in C. This means that C cannot be just a multiple of the total energy unless it vanishes identically. Finally, a rotation of coordinates can be performed to allow the elimination of ϵ unless $\epsilon^2+\gamma^2=0$, a case that Darboux also ignored. In the general case, Eq. (9) becomes with $\alpha=1$

$$xy(V_{xx}-V_{yy})+(y^2-x^2+\gamma)V_{xy}+3yV_x-3xV_y=0. \tag{10}$$

The solution of Eq. (10) was obtained by Darboux[1] as

$$V=f(u)-g(v)/(u^2-v^2), \tag{11}$$

where u and v are given by

$$2u^2=\rho^2+\gamma+[(\rho^2+\gamma)^2-4\gamma x^2]^{1/2},$$
$$2v^2=\rho^2+\gamma-[(\rho^2+\gamma)^2-4\gamma x^2]^{1/2},$$
$$\rho^2=x^2+y^2,$$

and f and g are two arbitrary functions.

In the degenerate case $\gamma=0$, the singular limit of the solution is

$$V=G(\rho)+\Phi(y/x)/\rho^2, \tag{12}$$

where G and Φ are also two arbitrary functions. If $\epsilon^2+\gamma^2=0$, Eq. (9) becomes

$$(xy\pm i\gamma/2)(V_{xx}-V_{yy})+(y^2-x^2+\gamma)V_{xy}+3yV_x-3xV_y=0,$$

the general solution of which is

$$V=\{F[\rho^2+(\rho^4-2\gamma z^2)^{1/2}]+G[\rho^2-(\rho^4-2\gamma z^2)^{1/2}]\}/(\rho^4-2\gamma z)^{1/2}, \tag{13}$$

with $z=x\pm iy$, and where F and G are again two arbitrary functions. In the degenerate case $\gamma=0$, we recover the singular limit (12).

The case *not* analyzed by Darboux corresponds to $\alpha=0$. Before proceeding to examine this case in detail, let us deal with the particular case where, in addition, β and δ also vanish. One obtains from (9), with $\gamma=1$,

$$V_{xy}+\epsilon(V_{xx}-V_{yy})=0.$$

If $4\epsilon^2+1\neq 0$, the solution of this equation is straightforward. An adequate rotation reduces it to

$$V_{xy}=0,$$

which indicates a separable potential

$$V=f(x)+g(y), \tag{14}$$

where f and g are two arbitrary functions. If $\epsilon=\pm i/2$, this rotation becomes singular, the potential is not separable, and the general solution is

$$V=(x^2+y^2)F(x\pm iy)+G(x\pm iy), \tag{15}$$

where F and G are again two arbitrary functions. This potential will be called quasi separable.

We now turn to the case $\alpha=0$ but β (or δ) nonvanishing. Translations of x and y allow us to eliminate $\gamma-\zeta$ and ϵ. Putting $\lambda=-\delta/\beta$, we obtain

$$2yV_{xy}+3V_x+x(V_{xx}-V_{yy})+\lambda[2xV_{xy}+3V_y-y(V_{xx}-V_{yy})]=0. \tag{16}$$

Finally an adequate rotation of coordinates allows the choice $\lambda=0$, which eliminates $\dot{x}^2$ in the constant of motion, unless $\lambda^2+1=0$, where this rotation becomes singular. (The choice $\lambda=\infty$ would correspond to eliminating $\dot{y}^2$ in the constant of motion). The case $\lambda^2+1=0$ will be treated in Sec. IV. In the general case, we obtain

$$2yV_{xy}+3V_x+x(V_{xx}-V_{yy})=0. \tag{17}$$

Several solutions to this PDE can be found by inspection, and the general solution will be presented in Sec. IV. One important class of solutions of Eq. (17) are the homogeneous polynomial solutions. They can be derived in an elementary way and turn out to be

$$V_0=1,\quad V_1=2y,\quad V_2=4y^2+x^2,$$
$$V_3=8y^3+4x^2y,\quad V_4=16y^4+12x^2y^2+x^4, \tag{18}$$
$$V_5=32y^5+32x^2y^3+6x^4y,\quad \text{etc.}$$

As we explained in Ref. 6, the mere existence of the quintic polynomial compelled us to weaken the Painlevé property criterion of integrability.

III. PAINLEVÉ ANALYSIS OF EQUATIONS OF MOTION

Let us consider the equations of motion associated with the integrable polynomial potential $-V_5/2^5$ [cf. Eq. (18)]:

$$\ddot{x} = 2y^3x + \tfrac{3}{4}yx^3,$$
$$\ddot{y} = 5y^4 + 3y^2x^2 + \tfrac{3}{16}x^4. \tag{19}$$

A leading order analysis in the complex t plane shows that a possible behavior of the solution near a singularity is

$$x,y \propto (t - t_0)^{-2/3}.$$

This, in itself, is *not* yet incompatible with the original conjecture as x^3 and y^3 could have been pure poles. However, the study of the resonances[7] shows that this is not the case. There is one resonance, namely $\frac{10}{3}$, which is positive and not an integer. This shows that $(t - t_0)^{2/3}x$ has in general an algebraic singularity. However, the expansion of x and y does not contain anything "worse" than powers of $(t - t_0)^{1/3}$, which is *still* quite remarkable.

The fact that the occurrence of such "natural" algebraic singularities does not compromise integrability suggests a generalization of the ARS conjecture[7] for two-dimensional systems. We will say that a system has the "weak-Painlevé property" whenever the solution, in the neighborhood of a singularity at t_0, can be expressed as an expansion in powers of $(t - t_0)^{1/r}$, where r is an integer to be defined below. The new conjecture is now that two-dimensional integrable systems possess the weak-Painlevé property.

In order to define the exponent $1/r$, let us consider a polynomial potential of degree $p + 2$. The Painlevé analysis proceeds in two steps. First, one studies the homogeneous, highest order, part of the potential. Whenever the Painlevé property is satisfied, one looks for lower order terms that do not destroy it. Incidentally, the same two steps procedure is used when one attempts a direct calculation of the integrals of motion.

Thus, to begin with, we restrict ourselves to a homogeneous polynomial potential of degree $p + 2$. It is obvious that the leading behavior in the neighborhood of a singularity will be

$$x,y \propto (t - t_0)^{-2/p}.$$

Moreover, there is *always* a resonance at $2 + 4/p$, which, except for $p = 1$, 2, and 4, is *not* an integer. Thus, there is no hope for $(t - t_0)^{2/p}x$ to be regular. In general, the solution will be an expansion in powers of $(t - t_0)^{1/p}$, which shows that one can take $r = p$. Indeed, further studies[10] have revealed a need to refine the definition of the "natural" power r. In this paper, however, the choice $r = p$ is sufficient, as our aim here is to study a specific family of "weak-Painlevé" potentials. In the following paper,[10] we will deal with the exhaustive search of weak-Painlevé integrable polynomials of degree 3. Since the investigation is more and more cumbersome with increasing order, we do not try to be exhaustive in the present work, but to show how useful the heuristic power of the weak-Painlevé property is.

Let V be a homogeneous polynomial of degree $p + 2$. We do not lose much generality by assuming that through a suitable rotation, the coefficient of xy^{p+1} can be set to zero while that of y^{p+2} does not vanish. We write V as

$$V = -\sum_{k=0}^{p+2} a_k x^k y^{p+2-k},$$

with $a_1 = 0$. The equations of motion are

$$\ddot{y} = -V_y = \sum_{k=0}^{p+1} (p + 2 - k)a_k x^k y^{p+1-k}, \tag{20}$$

$$\ddot{x} = -V_x = \sum_{k=2}^{p+2} k a_k x^{k-1} y^{p+2-k}. \tag{21}$$

In general, there are singularities in the neighborhood of which both x and y behave as $(t - t_0)^{-2/p}$. In addition, there are singularities near which y still behaves as $(t - t_0)^{-2/p}$, while x need not diverge that fast. Indeed, since a_1 vanishes, the right-hand side of Eq. (21) need not diverge as $(t - t_0)^{-2-2/p}$ but only as $x(t - t_0)^{-2}$, whatever the behavior of x is, provided it is *less* singular than that of y. Whenever it happens that $a_{p+1} = 0$ while $a_{p+2} \neq 0$, there will also be a singularity where x behaves as $(t - t_0)^{-2/p}$ while y does not diverge that fast. We must keep this possibility in mind, but we will *not* treat x and y on the same footing as the rotation has been explicitly chosen in order to have $a_1 = 0$.

Let us first consider the singularity in the neighborhood of which y diverges as $(t - t_0)^{-2/p}$ but not x. Then Eq. (20) fixes the coefficient of the leading term of y. We write

$$y = b(t - t_0)^{-2/p} + \epsilon(t - t_0)^{-2/p},$$

where $\epsilon \to 0$ as $t \to t_0$, and by equating the fastest diverging terms we find

$$2(p + 2)b/p^2 = (p + 2)a_0 b^{p+1}.$$

Since y must actually diverge as $(t - t_0)^{-2/p}$, $b \neq 0$ and thus

$$a_0 b^p = 2/p^2.$$

The possible behaviors for x are determined by Eq. (21). Let γ be the power dependence of x, i.e.,

$$x \propto (t - t_0)^{\gamma},$$

we find by equating the fastest diverging terms

$$\gamma(\gamma - 1) = 2a_2 b^p.$$

A first necessary condition for the weak-Painlevé property to be satisfied is that $\gamma = u/p$ with u an integer (positive or negative). This gives the equation

$$4a_2/a_0 = u(u - p). \tag{22}$$

One must now compute the resonances.[7] This is rather easy since the problem separates into resonances for y, which are -1 and $2 + 4/p$, and resonance for x which are 0 and $1 - 2u/p$. We do not get any further conditions since we already need u to be an integer. The only powers of $(t - t_0)$ that will appear at the resonances are multiples of $1/p$. We have not yet shown that no logarithms will get into the picture, but we are free of "nonnatural" powers of $(t - t_0)$, for this type of singularity, at least.

We note that -1 corresponds to the arbitrariness of t_0

and 0 to the arbitrariness of the coefficient of $(t-t_0)^{u/p}$ in x. The equation for a resonance n of y is only in terms of the quantity $N=(n-2/p)(n-1-2/p)$. Since $n=-1$ is always a resonance, it follows that $N_1=(1+2p)(2+2/p)$ satisfies the resonance condition; hence $2+4/p$ is also always a resonance as we noted above. On the other hand, the possible behaviors for x are $(t-t_0)^{u/p}$, where u satisfies Eq. (22). There are in general two roots to this equation, related by

$$u'+u''=p.$$

Note that at least one of these roots is positive, thus strictly larger than -2, and the corresponding behavior of x is less divergent than that of y. The fact that if x behaves as $(t-t_0)^{u'/p}$, the resonance is $1-2u'/p$ expresses that

$$u''/p=1-u'/p=u'/p+(1-2u'/p),$$

i.e., this resonance corresponds to the freedom of the coefficient of $(t-t_0)^{u'/p}$ in x. In fact, this resonance is only actually present if

$$u''>u'>-2;$$

otherwise, it is purely formal.

Let us now turn to the case where both x and y diverge as $(t-t_0)^{-2/p}$, i.e.,

$$x=\alpha(t-t_0)^{-2/p}+\epsilon_1(t-t_0)^{-2/p},$$
$$y=\beta(t-t_0)^{-2/p}+\epsilon_2(t-t_0)^{-2/p}.$$

where ϵ_1 and ϵ_2 go to zero as $t\to t_0$. We choose the following notation: Whenever V or one of its partial derivatives has an overbar, it means that it is estimated at $x=\alpha$, $y=\beta$. Then we get the following equations for α and β, by equating the most diverging terms in Eq. (20) and (21):

$$\begin{aligned}2(2+p)\alpha/p^2&=-\overline{V}_x,\\ 2(2+p)\beta/p^2&=-\overline{V}_y.\end{aligned}\tag{23}$$

In general, this system can have several couples (α,β) as solutions. For a given choice of (α,β) that satisfies Eqs. (23), a resonance n will occur whenever the determinant of the matrix M vanishes, where M is given by

$$M=\begin{pmatrix}(n-2/p)(n-1-2/p)+\overline{V}_{xx} & \overline{V}_{xy}\\ \overline{V}_{xy} & (n-2/p)(n-1-2/p)+\overline{V}_{yy}\end{pmatrix}.$$

Note that n only enters through the expression $N=(n-2/p)(n-1-2/p)$. Again, $n=-1$ is always a resonance; thus $N_1=(1+2/p)(2+2/p)$ must be a solution, as can be checked directly. Indeed, since V_x is a homogeneous polynomial of degree $p+1$,

$$xV_{xx}+yV_{xy}=(p+1)V_x.$$

Thus, for any couple (α,β) that satisfies Eqs. (23),

$$\alpha\overline{V}_{xx}+\beta\overline{V}_{xy}=-2(p+2)(p+1)\alpha/p^2=N_1\alpha,$$

and similarly

$$\alpha\overline{V}_{xy}+\beta\overline{V}_{yy}=-2(p+2)(p+1)\beta/p^2=-N_1\beta.$$

With the choice $N=N_1$, the matrix M becomes

$$M=\begin{pmatrix}-(\alpha/\beta)\overline{V}_{xy} & \overline{V}_{xy}\\ \overline{V}_{xy} & -(\beta/\alpha)\overline{V}_{xy}\end{pmatrix},$$

and its determinant indeed vanishes. This shows that $2+4/p$ is also always a resonance as it leads to the same value N_1 for N. The other value N_2 of N for which the determinant vanishes obviously satisfies

$$N_2+N_1=-\overline{V}_{xx}-\overline{V}_{yy}.$$

A second necessary condition for the weak-Painlevé property to be satisfied is that N_2 leads to resonances n that are of the form s/p, with s an integer. This can be written

$$-p^2(\overline{V}_{xx}+\overline{V}_{yy})=(s-2)(s-2-p)+(2+p)(2+2p).\tag{24}$$

This equation must be satisfied for some integer s for each choice of a couple (α,β) that solves Eqs. (23). The integer s could very well depend on the choice of the couple (α,β). It is in general difficult to find *all* the solutions to this problem, even for small values of p. But one can find *some* solutions by using a simplifying assumption. Since $a_1=0$, V_x has no term proportional to y^{p+1}, and x thus can be factored out of V_x. Then V_x/x is a homogeneous polynomial in x and y of degree p, which is also true of $V_{xx}+V_{yy}$. The simplifying assumption we will make is that these polynomials are proportional, their ratio being chosen in such a way as to satisfy Eq. (24). This is a very strong restriction. It implies that the resonances are the same for all choices of the couple (α,β), and this is certainly not a necessary condition. Still, it will prove very rich though not exhaustive. If we write

$$V_{xx}+V_{yy}=\lambda V_x/x,\tag{25}$$

Eq. (24) becomes

$$-\lambda p^2\overline{V}_x/\alpha=(s-2)(s-2-p)+(2+2p),$$

and since α and β satisfy Eqs. (23)

$$2\lambda(2+p)=(s-2)(s-2-p)+(2+p)(2+2p).\tag{26}$$

Equation (25) leads to a recursion relation between a_{k+2} and a_k. This recursion relation is

$$\begin{aligned}&(k+2)(k+1)a_{k+2}+(p+2-k)(p+1-k)a_k\\ &\quad=\lambda(k+2)a_{k+2}.\end{aligned}\tag{27}$$

If λ is a positive odd integer strictly smaller than $p+2$, there are no solutions with $a_0\neq 0$, because the coefficient of $a_{\lambda+1}$ will vanish. Such values of λ must be discarded. In general, only a's with even indices do not vanish, unless λ is a nonzero even integer strictly smaller than $p+2$, in which case $a_{\lambda+1}$ need not vanish, although $a_{\lambda-1}=0$. Thus the polynomial V is entirely determined by λ, up to the multiplicative arbitrary constant a_0 or, in the special case above, by two arbitrary constants a_0 and $a_{\lambda+1}$.

From Eq. (27), the ratio a_2/a_0 is given by

$$a_2/a_0=(p+2)(p+1)/2(\lambda-1).$$

Comparing with Eq. (22), we see that λ must satisfy

$$\lambda=1+2(p+2)(p+1)/[u(u-p)],\tag{28}$$

with u an integer. The problem is now to find two integers u

and s that satisfy simultaneously Eqs. (26) and (28).

If p is odd, a_{p+1} does not vanish, and there is no singularity where x diverges faster than y. If p is even, on the other hand, a_{p+1} in general vanishes, and there will be such a singularity, unless

$$2 \leqslant \lambda \text{ (even)} < p + 2. \tag{29}$$

A necessary condition for this singularity to be of the weak-Painlevé type is

$$4a_p/a_{p+2} = v(v-p),$$

with v an integer, in analogy to Eq. (22). From Eq. (27), this can be written

$$v(v-p) = 2(p+2)(\lambda - p - 1). \tag{30}$$

If p is even, one must thus solve (26), (28), and (29) or (30), while (26) and (28) suffice for p odd.

For every p there are always at least two solutions to this problem [note that solutions depend only on $(s-2)$ $(s-2-p)$ and $u(u-p)$, $v(v-p)$]:

(i) $u = p + 2$ (or -2), $s = 0$ (or $p+4$),

with $\lambda = p + 2$,

(ii) $u = p + 1$ (or -1), $s = 2p + 6$ (or $-p-2$),

with $\lambda = 2p + 5$,

as can easily be checked. These solutions, for even p, do lead to integer values of v, $p+2$ (or -2), and $2p+4$ (or $-p-4$), respectively.

There might be other solutions for some values of p, but none were found for small values of p except $p = 1$, $u = 4$ (or -3), $s = 2$ (or 3), with $\lambda = 2$. This leads to the following potential:

$$V = a_0(y^3 + 3yx^2) + a_3x^3,$$

which is known to be integrable (separable).

Let us now consider the first solution. In that case there is a resonance $n = s/p$ equal to 0. This should mean that one of the coefficients α and β is arbitrary. Now, for even p, this is true. Substituting $\lambda = p + 2$ into Eq. (27) allows us to compute V, and one finds

$$V = a_0(y^2 + x^2)^{p/2}.$$

Thus only $(\alpha^2 + \beta^2)$ is determined and the occurrence of the resonance $n = 0$ is normal. This potential is obviously integrable. This case, however, is not as trivial as it seems, because one can now add a lower degree polynomial that need *not* be rotationally invariant, without destroying integrability. One thus recovers exactly the polynomial potentials of the family found by Darboux,[1] which are indeed all even. For p odd, on the other hand, the resonance $n = 0$ is pathological, as it is not related to the arbitrariness of either α or β. If one solves Eqs. (20) and (21), one finds

$$\alpha = 0, \quad \beta = b.$$

This is not at all acceptable, as α may not vanish, since we assumed that x does behave as $(t - t_0)^{-2/p}$. Although we found a "formal" solution to the arithmetical equations, we do not have in this case a weak-Painlevé potential. In fact, the vanishing of α entails here a logarithmic singularity. For $p = 1$, this "pseudo-Painlevé" case has been noted before.[11]

The new and interesting case is

$$u = -1 \text{ (or } p+1), \quad n = 2p + 6, \text{ (or } -p-2),$$
with $\lambda = 2p + 5$,

complemented by $v = 2p + 4$ (or $-p-4$) for even p. One can solve the recursion relation Eq. (27) with $a_0 = 2^{p+2}$ to find

$$a_{2k} = 2^{p+2-2k} c_{p-k}^k,$$

for $0 \leqslant 2k \leqslant p + 2$, $a_{2k+1} = 0$.

This family of polynomials are indeed integrable. Note that, here, these homogeneous potentials are already nontrivial. They coincide with the polynomials of Eq. (18). To see that, we rewrite Eq. (25) as

$$x(V_{xx} + V_{yy}) = (2p+5)V_x,$$

and note that V_x is homogeneous of degree $(p+1)$, i.e.,

$$(p+1)V_x = xV_{xx} + V_{xy};$$

thus one recovers Eq. (17),

$$x(V_{yy} - V_{xx}) - 2yV_{xy} = 3V_x.$$

The homogeneous polynomials we have found satisfy a set of necessary conditions for the weak-Painlevé property to be true. To check that they are sufficient, one should ascertain that no logarithms appear at the resonances. This had indeed been found to hold for some low values of p. It would be straightforward, although increasingly tedious, to verify this property for higher values of p. It will be even more cumbersome, except for very small values of p, to check that the only lower degree terms one can add without destroying the weak-Painlevé properly are the polynomials that solve the same Eq. (17). Actually these two properties (absence of logarithms and superposition of solutions) are expected to hold for all p, because Eq. (17) is just the integrability condition. There is no need to prove that rigourously. As a matter of fact, the ARS conjecture is meant to be a heuristic tool rather than a rigorous approach. In the case at hand it has well served its purpose by leading to the same integrability condition (17) as the direct approach.

IV. SOLUTION OF THE PDE TO BE SATISFIED BY THE POTENTIAL

The solutions of Eq. (17) can be obtained in a variety of ways (Fourier transform, use of characteristics, etc.). However, given the particular form of the equation and the intuition stemming from the knowledge of particular solutions, a direct approach appears simpler. We start by introducing a new set of variables

$$\rho = (x^2 + y^2)^{1/2}, \quad \eta = y. \tag{31}$$

A straightforward calculation obtains

$$V_{\rho\rho} + 2V_\rho/\rho - V_{\eta\eta} = 0. \tag{32}$$

We do away with the first-order derivative by introducing

$$U = \rho V.$$

There results the equation

$$U_{\rho\rho} - U_{\eta\eta} = 0, \tag{33}$$

which is the well known form of the wave equation. Its solu-

tion reads

$$U = f(\rho + \eta) + g(\rho - \eta),$$

with f and g arbitrary functions. So the solution for the potential V becomes

$$V = [f(\rho + \eta) + g(\rho - \eta)]/\rho. \tag{34}$$

One can easily check from this form that this family of integrable potentials cannot be contained in the Darboux solution (11), even though the intersection of the two sets is not empty. Once the form of the potential is obtained, we can integrate Eqs. (7) and deduce the form of the integral of motion. We start by rewriting Eqs. (7) with respect to the variables ρ and η:

$$h_\rho = \eta V_\rho - \rho V_\eta, \quad h_\eta = \eta V_\eta - \rho V_\rho.$$

The integration is straightforward once one expresses V through Eq. (34), and we thus find

$$h = [(\rho + \eta)g(\rho - \eta) - (\rho - \eta)f(\rho + \eta)]/\rho. \tag{35}$$

The same results Eqs. (34) and (35) would have been obtained in a more systematic (and tedious) way by looking for the characteristics of Eq. (17), which in view of Eqs. (31), define a system of parabolic coordinates.

In the special case $\lambda^2 + 1 = 0$, Eq. (16) becomes

$$2\bar{z}V_{\bar{z}\bar{z}} + 3V_{\bar{z}} = 0,$$

with $z = x \pm iy$, $\bar{z} = x \mp iy$. The general solution is

$$V = F(x \pm iy)/(x^2 + y^2)^{1/2} + G(x \pm iy), \tag{36}$$

where F and G are two arbitrary functions.

With Eqs. (11)–(15), the results (34) and (36) complete the answer to the question of the existence of an integral of motion quadratic in velocities for the motion of a particle in two dimensions. It is, however, instructive to examine some particular families of solutions.

(a) Let us start with the potential

$$V = 1/\rho.$$

This potential is manifestly contained in Eq. (34), and at the same time, from Eq. (12), belongs to the Darboux family for $\gamma = 0$. Its Darboux integral is simply

$$C_1 = (x\dot{y} - y\dot{x})^2,$$

i.e., the square of the angular momentum. However from Eq. (35) another integral of motion can be computed

$$C_2 = \dot{y}\dot{x}^2 - x\dot{x}\dot{y} + y/\rho$$

or

$$C_2 = \dot{x}(x\dot{y} - y\dot{x}) + y/(x^2 + y^2)^{1/2}.$$

This just the y component of the quantity

$$\dot{\mathbf{r}} \times \mathbf{L} + \mathbf{r}/|\mathbf{r}|,$$

which is the third integral of motion in the case of the Kepler problem. The existence of an additional integral in this case is associated with the degeneracy of the problem, which is manifested, for instance, by the separability in more than one system of coordinates. From a physical point of view, the usefulness of this invariant resides in the fact that it helps fix the orientation of the orbit in the plane.

(b) Polynomial solutions: In Ref. 6, we already presented the homogeneous polynomial solutions of Eq. (17). Their general form is

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} C^k_{n-k} x^{2k} y^{n-2k}, \tag{37}$$

where

$$C^k_{n-k} = \binom{n-k}{k} = (n-k)!/k!(n-2k)!.$$

We can obtain them directly from Eq. (34) by choosing

$$f(q) = (-1)^n g(q) = q^{n+1}/2. \tag{38}$$

The general solution can be built from a superposition of the homogeneous polynomials of each degree, due to the linearity of Eq. (17). The simplest polynomial solution to Eq. (17) (apart from $V_0 = 1$) is, of course, $V_1/2 = y$. This means that a multiple of y can always be added to the potential. In the case of polynomial potentials this fact is quite useful as it allows us the freedom of a translation in the y direction without the appearance of a linear term in the potential. A polynomial potential case of particular interest is the Hénon–Heiles potential for which Greene[4] obtained the second integral of motion. In this case of integrability, the Hénon–Heiles potential reads

$$V = (Ax^2 + By^2)/2 + x^2 y + 2y^3.$$

It can be shown that this potential reduces to a superposition of the potentials V_3, V_2, and V_1 of Eq. (37) after the adequate translation.

(c) Homogeneous solutions: If we allow negative values of n in Eq. (38) one finds rational integrable potentials, which can be related to the polynomials V_n by

$$V_{-n-2} = (-1)^n V_n/x^{(2n+2)}.$$

In the case $n = -1$ one would have $f = -g = 1/2$ and V_{-1} vanishes. In addition to this family of homogeneous rational potentials there exists a second homogeneous family W_n given by $f(q) = (-1)^{n+1} g(q) = q^{n+1}/2$, which can be related between themselves by

$$W_{-n-2} = (-1)^{n+1} W_n/x^{(2n+2)}.$$

For $n \geqslant 0$, ρW_n is an homogeneous polynomial in x, y of degree $n + 1$, and W_{-1} is just $1/\rho$.

We thus find two potentials homogeneous of order n for all n, except $n = -1$ where we have as yet only one since $V_{-1} = 0$. Actually, a second potential homogeneous of order -1 exists and is

$$Z = \ln[(\rho + y)/(\rho - y)]/\rho,$$

or

$$Z = 2 \ln \tan(\tfrac{1}{2} \arctan y/x + \pi/4)/\rho.$$

This exhausts all homogeneous solutions of integer ($\gtrless 0$) order.

V. CONCLUSION AND OUTLOOK

The main result of this paper is the identification of a whole new family of integrable dynamical systems for one particle in two space dimensions. This was achieved through the derivation and resolution of a PDE to be satisfied by the potential in order to ensure the existence of an integral of motion quadratic in the velocities.

The most important point is that this equation was derived in two different ways. The first was a direct calculation based on the search for a constant of the motion. The second was through a suitable implementation of the "weak-Painlevé" property of the equations of motion, which we introduced in a previous publication.[6] This constitutes a first evidence of the usefulness of the weak-Painlevé concept.

In a companion paper[10] we will have the occasion to deal in more detail with this notion. We will actually show that the weak-Painlevé property is a very efficient tool for the identification of integrable cases. More precisely, we were able to find *all* weak-Painlevé case for third degree polynomial potentials, the integrability being confirmed in each case by direct computation of the integral of motion. In the case of fourth-degree polynomials, new cases of integrability will be presented. Finally we will focus on the precise definition of the weak-Painlevé property, and we will insist on its conditions of applicability and the unavoidable "caveat."

[1]G. Darboux, Arch. Néerl. (ii) **VI**, 371 (1901).
[2]J. Bertrand, J. Math. (i) **XVII**, 121 (1852).
[3]E. T. Whittaker, *Analytical Dynamics of Particles* (Cambridge U.P., Cambridge, 1959), 4th ed., pp. 334–35 (1st ed. 1904).
[4]J. Greene, quoted in Y. F. Chang, M. Tabor, and J. Weiss, J. Math. Phys. **23**, 531 (1982).
[5]L. J. Hall, Preprint L. L. L., UCID-18980 Rev. 1 (1982), submitted to Ann. Phys. (N.Y.).
[6]A. Ramani, B. Dorizzi, and B. Grammaticos, Phys. Rev. Lett. **49**, 1539 (1982).
[7]M. J. Ablowitz, A. Ramani, and H. Segur, Lett. Nuovo Cimento **23**, 333 (1978); M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. **21**, 715 (1980).
[8]J. Moser, Adv. Math. **16**, 197 (1975).
[9]M. Hénon, Phys. Rev. B **9**, 1921 (1974).
[10]B. Grammaticos, B. Dorizzi, and A. Ramani, J. Math. Phys. **24**, 2289 (1983).
[11]T. Bountis and H. Segur, in *The Proceedings of the Workshop on "Mathematical Systems"* edited by M. Tabor and Y. Trève, AIP Conference Proceedings, Vol. 88 (AIP, New York, 1982), p. 279.

Integrability of Hamiltonians with third- and fourth-degree polynomial potentials[a)]

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The weak-Painlevé property, as a criterion of integrability, is applied to the case of simple Hamiltonians describing the motion of a particle in two-dimensional polynomial potentials of degree three and four. This allows a complete identification of all the integrable cases of cubic potentials. In the case of quartic potentials, although our results are not exhaustive, some new integrable cases are discovered. In both cases the integrability is explicited by a direct calculation of the second integral of motion of the system.

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I. INTRODUCTION

In this paper we carry on our investigations on the integrability of dynamical systems based on the generalization of the Painlevé criterion we introduced in Ref. 1. In systems with more than one degree of freedom integrability is of rare occurrence. (Integrability, in the case of Hamiltonian systems, is defined as the existence of analytical, single-valued integrals of motion, in number equal to that of the degrees of freedom of the system, time independent and in involution.) The detection of integrability has been greatly facilitated these last years by the work of Ablowitz, Ramani, and Segur.[2] They have conjectured, and verified in a multitude of cases of physical interest, that integrability is intimately related to the analytic properties of the solutions of the equations of motion. Namely, whenever the solutions possess the Painlevé property, i.e., their only movable singularities on the complex time plane are poles, the system is integrable. The reciprocal is also true, the Painlevé property having been verified for the known integrable systems (eventually after some, trivial or not, changes of variables). However, although the ARS conjecture holds true for many (or infinite) degrees of freedom systems, it became clear from the results presented in Ref. 1 that some revision of the Painlevé criterion was needed for two-dimensional systems. (Actually, for one-dimensional systems, the Painlevé criterion is superfluous: They are integrable by definition). We discovered, namely, a quintic polynomial Hamiltonian in two dimensions, which did not possess the Painlevé property in the classical manner while being integrable. The integrability was ensured by the explicit calculation of the second integral of motion. However, some regularity still persisted: The solution in the neighborhood of a singularity could be expressed as an expansion in powers of $(t-t_0)^{1/3}$. This led us to the introduction of the "weak Painlevé" notion as a criterion of integrability. The "weak Painlevé property" was associated to an expansion in powers of $(t-t_0)^{1/r}$, with r being a "natural" power, determined solely by the dominant terms in the equations of motion.

In the preceding paper, Ref. 3, we presented a first application of the weak Painlevé concept. A partial differential equation, obeyed by a certain class of integrable potentials, was derived, based on the weak Painlevé property, which we ensured in the easiest way through some simplifying assumptions. The same equation was derived through a direct search for integrals of motion quadratic in velocities, and its solution provided a whole new class of integrable systems, to which the quintic polynomial, mentioned above, belongs.

This twofold approach, integrability prognostication based on the Painlevé criterion, and integrability verification through a direct calculation of the constants of motion is pursued in the present work as well. By deleting some of the simplifying assumptions we had adopted in Ref. 3, we are able to extend our study of integrability to a wider class of potentials. However, as the calculations become very rapidly prohibitively intricate, one is compelled to restrict oneself to some low-order polynomial potentials. The aim of the present paper is to present our results on third and fourth degree polynomial potentials. The direct method for the search of integrals of motion is first introduced in Sec. II. Sections III and IV are devoted respectively to the study of third and fourth degree Hamiltonians. Section V finally deals with the definition of the "natural" power we alluded to previously, definition which is the cornerstone of the "weak Painlevé" property.

II. DIRECT SEARCH FOR INTEGRALS OF MOTION

In Ref. 3 we presented a general method for the investigation of the existence of integrals of motion quadratic in the velocities, following Bertrand's approach.[4] In the present work we are going to focus on integrals quartic in the velocities. For the sake of completeness we start by sketching briefly the case of a constant of motion cubic in velocities.

We start with a form:

$$C = f_0\dot{x}^3 + f_1\dot{x}^2\dot{y} + f_2\dot{x}\dot{y}^2 + f_3\dot{y}^3 + g_0\dot{x} + g_1\dot{y}. \tag{2.1}$$

[a)] The authors dedicate this work to their Soviet colleague, the mathematician and physicist Salomon Alber.

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No even power in the velocities terms are allowed, as we suppose that the Hamiltonian is invariant under time reversal. The condition $dC/dt = 0$ leads to a system of partial differential equations obtained by equating to zero the coefficients of each monomial $\dot{x}^m\dot{y}^n$. We thus obtain

$$\frac{\partial f_0}{\partial x} = 0, \quad \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial x} = 0, \quad \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0,$$
$$\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x} = 0, \quad \frac{\partial f_3}{\partial y} = 0. \tag{2.2}$$

The solution to Eq. (2.2) is straightforward. The f_i's are cubic polynomials in (x,y):

$$\begin{aligned} f_0 &= \alpha y^3 + \beta y^2 + \gamma y + \delta, \\ f_1 &= -(3\alpha y^2 + 2\beta y + \gamma)x + \epsilon y^2 + \zeta y + \eta, \\ f_2 &= (3\alpha y + \beta)x^2 - (2\epsilon y + \zeta)x + \theta y + \kappa, \\ f_3 &= -\alpha x^3 + \epsilon x^2 - \theta x + \lambda. \end{aligned} \tag{2.3}$$

The next set of equations reads

$$\begin{aligned} &3f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0, \\ &2f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0, \\ &f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} = 0. \end{aligned} \tag{2.4}$$

One replaces f_i from (2.3) and $\ddot{x},\ddot{y}$ from the equations of motion, $\ddot{x} = -\partial V/\partial x$, $\ddot{y} = -\partial V/\partial y$, and integrates for g_i. The compatibility condition for the integration to be possible reads

$$\frac{\partial^2}{\partial x^2}(f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^2}{\partial x \partial y}(2f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^2}{\partial y^2}(3f_0\ddot{x} + f_1\ddot{y}) = 0. \tag{2.5}$$

Once the g_i are obtained there remains a last equation to be verified:

$$g_0\ddot{x} + g_1\ddot{y} = 0, \tag{2.6}$$

which, in view of (2.4), is a nonlinear PDE for the potential V. So, for the system to possess an integral of motion cubic in velocities, the potential must satisfy Eqs. (2.5) and (2.6). This actually can be realized in some cases, as, for example, the Toda system,[5,6] and will be the object of a future work. However, none of the systems examined in this paper fall into this class and we are thus led to consider integrals of motion quartic in velocities.

Let us look for an integral of motion of the form:

$$C = f_0\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 + g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h. \tag{2.7}$$

Again the first set of partial differential equations for f_i can be solved in a straightforward way giving:

$$\begin{aligned} f_0 &= \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \epsilon, \\ f_1 &= -(4\alpha y^3 + 3\beta y^2 + 2\gamma y + \delta)x + \zeta y^3 + \eta y^2 + \theta y + \rho, \\ f_2 &= (6\alpha y^2 + 3\beta y + \gamma)x^2 - (3\zeta y^2 + 2\eta y + \theta)x + \kappa y^2 + \lambda y + \mu, \\ f_3 &= -(4\alpha y + \beta)x^3 + (3\zeta y + \eta)x^2 - (2\kappa y + \lambda)x + \nu y + \xi, \\ f_4 &= \alpha x^4 - \zeta x^3 + \kappa x^2 - \nu x + \sigma. \end{aligned} \tag{2.8}$$

As we have explained in Ref. 3, the Hamiltonian, being a constant of motion, can be freely added to C. The same applies to the square of the Hamiltonian as well, which allows us one of the three choices $\epsilon = 0$ or $\mu = 0$ or $\sigma = 0$.

The PDE's for g_i read

$$\begin{aligned} &4f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} = 0, \\ &3f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_0}{\partial y} + \frac{\partial g_1}{\partial x} = 0, \\ &2f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} = 0, \\ &f_3\ddot{x} + 4f_4\ddot{y} + \frac{\partial g_2}{\partial y} = 0. \end{aligned} \tag{2.9}$$

As in the case of cubic integrals, for Eqs. (2.9) to be integrable, the potential must satisfy a compatibility condition:

$$-\frac{\partial^3}{\partial x^3}(f_3\ddot{x} + 4f_4\ddot{y}) + \frac{\partial^3}{\partial x^2 \partial y}(2f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^3}{\partial x \partial y^2}(3f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^3}{\partial y^3}(4f_0\ddot{x} + f_1\ddot{y}) = 0. \tag{2.10}$$

Next, the equations for h are obtained:

$$\begin{aligned} &2g_0\ddot{x} + g_1\ddot{y} + \frac{\partial h}{\partial x} = 0, \\ &g_1\ddot{x} + 2g_2\ddot{y} + \frac{\partial h}{\partial y} = 0. \end{aligned} \tag{2.11}$$

The compatibility condition for this last system reads:

$$\frac{\partial}{\partial y}(2g_0\ddot{x} + g_1\ddot{y}) = \frac{\partial}{\partial x}(g_1\ddot{x} + 2g_2\ddot{y}). \tag{2.12}$$

So an integral of motion quartic in velocities exists whenever the potential satisfies the PDE's (2.10) and (2.12), the latter being a nonlinear one. The search is somewhat facilitated by the fact that in all nontrivial known cases of integrable potentials the f_i are just constants, independent of x and y. In that case equation (2.10) [and (2.5) as well] can be easily solved. As a matter of fact, for third degree polynomial potentials, Eq. (2.10) is automatically satisfied with constant f_i's while, for fourth degree potentials, Eq. (2.10) amounts to just one constraint.

III. THIRD DEGREE POLYNOMIAL POTENTIALS

As we have explained in the previous section, the integrability conditions for the motion of a particle in a two-dimensional potential take the form of a system of two PDE's, one of which is nonlinear. The search for the general solution of this system would have been an unrealistic attempt, given the complexity of the problem. One can look, however, at cases of physical interest with potentials of form simple enough for the problem to be tractable.

In this section we will examine in detail the case of a homogeneous polynomial potential of third degree. Our Hamiltonian can thus be viewed as a generalization of the Hénon–Heiles one.[7] The general form of the potential (within a global multiplicative factor which can be absorbed by a proper rescaling of time or space) is

$$V = y^3 + ay^2x + bx^2y + cx^3. \tag{3.1}$$

This form can be somewhat simplified through a rotation. It can be easily verified that one can almost always find a rotation which allows one to eliminate the y^2x term while preserving the y^3 one. There exist just two exceptions:

$$V = (x \pm iy)^3$$

and

$$V = (x^2 + y^2)(x \pm iy). \tag{3.2}$$

Incidentally, both potentials (3.2) are integrable. The first is manifestly separable while the second belongs to the quasiseparable case we introduced in Eq. (15) of Ref. 3. Both possess a second integral of motion quadratic in velocities. From now on, we limit ourselves to potentials of the form

$$V = y^3 + bx^2y + cx^3. \tag{3.3}$$

A. Painlevé analysis of the equations of motion

We will very closely follow the calculations of Ref. 3. The equations of motion are

$$\ddot{y} = -3y^2 - bx^2, \tag{3.4}$$

$$\ddot{x} = -2bxy - 3cx^2. \tag{3.5}$$

There are singularities in the neighborhood of which x and y both behave as $(t - t_0)^{-2}$. In addition, there are singularities where y behaves as $(t - t_0)^{-2}$ but x does not diverge that fast. Let us first consider the latter case. Equation (3.4) fixes the coefficient of the leading term of y. We write

$$y = \gamma(t - t_0)^{-2} + \epsilon(t - t_0)^{-2},$$

where $\epsilon \to 0$ as $t \to t_0$, and by equating the fastest diverging terms we find

$$6\gamma = -3\gamma^2.$$

Since y must actually diverge as $(t - t_0)^{-2}$, $\gamma \neq 0$, and thus

$$\gamma = -2.$$

The possible behaviors for x are determined by (3.5). Let s be the power dependence of x, i.e.

$$z \propto (t - t_0)^s;$$

we find, by equating the fastest diverging terms,

$$s(s - 1) = 4b. \tag{3.6}$$

We should demand that s be an integer because here weak Painlevé and Painlevé coincide. However, if $c = 0$, the equation for y contains only even powers of x, the equation for x contains only odd powers of x, and the system can be written in terms of y and x^2. In that case, the Painlevé property may be satisfied by x^2 rather than x, and this means that s may be a half-integer but *only* for $c = 0$. The search for resonances does not lead to any new condition (Ref. 3).

Let us now consider the case where both x and y diverge as $(t - t_0)^2$, i.e.,

$$x = \alpha(t - t_0)^{-2} + \epsilon_1(t - t_0)^{-2},$$

$$y = \beta(t - t_0)^{-2} + \epsilon_2(t - t_0)^{-2}.$$

where ϵ_1 and ϵ_2 go to zero as $t \to t_0$.

The equations for α and β are

$$6\beta = -3\beta^2 - b\alpha^2, \tag{3.7}$$

$$6\alpha = -2b\alpha\beta - 3c\alpha^2; \tag{3.8}$$

again α may not vanish as x actually diverges as $(t - t_0)^{-2}$. Thus (3.8) may be rewritten:

$$6 = -2b\beta - 3c\alpha. \tag{3.9}$$

In general, this system has two couples of solution (α,β). For each of them, a resonance n will occur whenever the determinant of the matrix M vanishes, where M is given by

$$M = \begin{pmatrix} (n-2)(n-3) + 2b\beta + 6c\alpha & +2b\alpha \\ +2b\alpha & (n-2)(n-3) + 6\beta \end{pmatrix}.$$

Note that n only enters through the expression $N = (n - 2)(n - 3)$. One solution is $N = 12$, which corresponds to $n = -1$ and $n = 6$. Indeed, using (3.7) and (3.9), the matrix M becomes

$$M = \begin{pmatrix} -2b\beta & 2b\alpha \\ 2b\alpha & -2b\alpha^2/\beta \end{pmatrix},$$

and its determinant clearly vanishes. The other solution N' for N satisfies:

$$N' + 12 = -(2b\beta + 6c\alpha + 6\beta).$$

Using once more Eq. (3.9), one finds

$$N' = (2b - 6)\beta. \tag{3.10}$$

The two choices of the couple (α,β) lead to two values of N'. Substituting α from Eq. (3.9) into (3.7), we find a second-order equation for β:

$$\beta^2(27c^2 + 4b^3) + \beta(54c^2 + 24b^2) + 36b = 0.$$

Its solutions β_1 and β_2 are such that

$$\beta_1 + \beta_2 = -\frac{54c^2 + 24b^2}{27c^2 + 4b^3} = -2 + \frac{(2b - 6)4b^2}{27c^2 + 4b^3},$$

$$\beta_1\beta_2 = \frac{36b}{27c^2 + 4b^3}.$$

The corresponding solutions N_1 and N_2 of (3.10) can be straightforwardly shown to satisfy

$$N_1 + N_2 = -2(2b - 6) + N_1N_2b/9. \tag{3.11}$$

This can be further symmetrized by remembering Eq. (3.6) and calling N_3 the quantity $s(s - 1)$. There results

$$36(N_1 + N_2 + N_3 - 12) = N_1N_2N_3. \tag{3.12}$$

A necessary condition for the Painlevé property to be satisfied is that N_1 and N_2 also be of the form $p_i(p_i - 1)$ with p_i integer and $i = 1$ or 2.

Remember that s should also be an integer except if $c = 0$, in which case s may be a half-integer. For any choice of s or equivalently of b, $N_1 = N_2 = 6$ is a solution of Eq. (3.11). However, this solution does not lead to a Painlevé potential but rather to a potential with logarithmic singularity which generalizes the well-known case of Ref. 8.

$$V = y^3 + 3x^2y/2.$$

Besides this formal solution the *only* other solutions of (3.12) up to the obvious permutations of N_1, N_2, N_3 are

$$N_3 = 0,\ N_1 = 0,\ N_2 = 12, \quad V = y^3 + \lambda x^3, \tag{3.13}$$

$$N_3 = \tfrac{3}{4},\ N_1 = 90,\ N_2 = 90, \quad V = y^3 + 3yx^2/16 (c = 0!), \tag{3.14}$$

$$N_3 = 2,\ N_1 = 30,\ N_2 = 30, \quad V = y^3 + yx^2/2, \tag{3.15}$$

$$N_3 = 2,\ N_1 = 20,\ N_2 = 90, \quad V = y^3 + yx^2/2 + ix^3/6\sqrt{3}. \tag{3.16}$$

Permutations of N_1, N_2, N_3 are equivalent to rotations that recover $a = 0$ where a is the coefficient of y^2x in the potential. Note that one could have, through permutations, N_1 or $N_2 = \frac{3}{4}$. In that case, expansions of both x and y contain half-integer powers of $(t - t_0)$ and superficially do not look Painlevé. However, the change of variables which recovers the Painlevé property is evident, namely the rotation back to the case $N_3 = \frac{3}{4}$ and the choice (y,x^2) instead of (y,x).

The case $N_3 = 12, N_1 = N_2 = 0$, is not completely equivalent to (3.13) because the rotation that leads from one case to another may be singular. The potential in that case writes

$$V = y^3 + 3x^2y + \mu x^3.$$

The appropriate choice of μ allows to recover all values of λ in (3.13) except $\lambda = 0$. Conversely, $\mu = \pm 2i$ do not correspond to any λ. These two conjugate potentials are not really separable. They enter in the quasiseparable class of Eq. (15) of Ref. 3. Potentials (3.14) and (3.15) have been identified by the La Jolla group.[9] The first has been integrated independently by Hall[10] and by Grammaticos, Dorizzi, and Padjen,[11] and the second has been integrated by Greene.[12]

The potential (3.16) is new. It is indeed integrable. The second integral of motion will be derived in the next subsection.

This exhausts *all* the Painlevé potentials of degree three.

B. Direct search for the integrals of motion

As we have explained in Sec. II, the integrals of motion for the potential (3.3) quartic in velocities correspond to constant f_i's. (It goes without saying that the case of general f_i's, as well as the case of cubic integrals, have been examined as well. However, we will not burden the presentation by exhibiting these calculations which did not yield any positive result.) With constant f_i's Eq. (2.10) is identically satisfied: $\ddot{x}$ and $\ddot{y}$ are quadratic in x, y. So one can integrate Eq. (2.9) for the g_i, which gives

$$\begin{aligned} g_0 &= f_1(3y^2x + x^3/3) + (2f_2 - 2f_2b/3 + 4f_4b/3 + f_3)y^3, \\ g_1 &= f_1[3cx^3 + 3(b-1)yx^2] + f_2(2bx^3/3 + 2by^2x) \\ &\quad + f_3[-3cy^2x + (3 - b/3)y^3] - 4f_4by^2x, \\ g_2 &= 4f_4(y^4 + byx^2) + f_3(3cyx^2 + by^2x) \\ &\quad + (f_3b + 2f_2c + f_1 - f_1b)x^3. \end{aligned} \tag{3.17}$$

Some integration constants have been taken equal to zero in (3.17), anticipating the results from the application of the compatibility condition (2.12). The latter applied to the g_i's of (3.17) leads to the following system, when one equates to zero the coefficient of each monomial $x^m y^n$:

$$\begin{aligned} &f_3(b-3)(b-45/2) = 0, \\ &10b^2f_3 + 30cbf_2 + (45c^2 + 13b - 43b^2/3)f_1 = 0, \\ &40b^2f_4 + 30cbf_3 + 4b(b-3)f_2/3 \\ &\qquad + (-90c + 60cb)f_1 = 0, \\ &-60bcf_4 + (-45c^2 + 7b^2 + 9b)f_3 + 30cbf_2 \\ &\qquad + f_1(45 - 99b + 18b^2) = 0, \\ &(b-3)(112bf_4/3 + 30f_3 - 56bf_2/3) = 0. \end{aligned} \tag{3.18}$$

The above system is linear in the f_i's but nonlinear in the coefficients b and c of the potential. The solution of this system is straightforward but quite tedious. Seven distinct (within a complex conjugation) solutions were found in all.

(a) $b = 0$, c free,

(b) $b = 3$, c free.

Those two solutions correspond to the separable potential (3.13), the two forms being equivalent within a rotation of coordinates which recovers $a = 0$ in the potential. This case generalizes the usual separable form of the Hénon–Heiles potential.[7]

(c) $b = 3/16$, $c = 0$,

(d) $b = 45/2$, $c = 178\sqrt{14}i$.

One recognizes in (c) the form (3.14), while case (d) corresponds to a rotation of (3.14).

(e) $b = \frac{1}{2}$, $c = i/6\sqrt{3}$,

(f) $b = 5$, $c = 22i$,

(g) $b = 45/2$, $c = 27\sqrt{3}/2i$.

These three cases correspond to (3.16) and its possible rotations. The only case which was not recovered in this analysis and necessitated the inclusion of quadratic terms in the f_i was the potential (3.15), which corresponds to potential parameters

(h) $b = \frac{1}{2}$, $c = 0$,

or, after a rotation

(i) $b = 15/2$, $c = 7i$.

The fourth order integral of cases (a,b) and (h,i) is trivial as it is just the square of an integral quadratic in the velocities.

On the contrary case (c,d) has a genuine fourth order integral:

$$C = \dot{x}^4 - \tfrac{3}{4}yx^2\dot{x}^2 + \tfrac{1}{4}x^3\dot{x}\dot{y} - \tfrac{3}{64}x^4y^2 - \tfrac{1}{128}x^6.$$

(Here we have preferred to take $f_4 = 0$ and $f_0 \neq 0$ in order to alleviate the notations.)

The same is true for the potential

$$V = y^3 + \tfrac{1}{2}x^2y + (i/6\sqrt{3})x^3.$$

Its second integral of motion reads:

$$\begin{aligned} C = {} & \dot{y}^4 + 2\dot{y}^2\dot{x}^2 - 2i\dot{y}\dot{x}^3/\sqrt{3} + (4y^3 + 2yx^2 - ix^3/3\sqrt{3})\dot{y}^2 \\ & + (i\sqrt{3}yx^2 + x^3)\dot{x}\dot{y} + (4y^3 - 2i\sqrt{3}y^2x - ix^3/3\sqrt{3})\dot{x}^2 \\ & + 4y^6 + 4y^4x^2 + ix^3y^3/3\sqrt{3} \\ & + 5y^2x^4/4 + ix^5y/6\sqrt{3} + x^6/54. \end{aligned}$$

So every case of integrability predicted by the Painlevé analysis was indeed recovered by the direct approach for the computation of the integrals of motion.

IV. FOURTH DEGREE POLYNOMIAL POTENTIALS

As in the previous section, we start by introducing a homogeneous polynomial potential, of degree four here:

$$V = y^4 + ay^3x + by^2x^2 + cyx^3 + dx^4, \tag{4.1}$$

and we perform a rotation in order to eliminate the term y^3x, while keeping the term y^4.

In analogy to the case of cubic potentials, this turns out to be possible unless one deals with the potential

$$V = (x \pm iy)^4 + \mu(x^2 + y^2)(x \pm iy)^2. \tag{4.2}$$

This is a potential of the quasiseparable class introduced in Eq. (15) of Ref. 3. It is integrable and possesses an integral quadratic in velocities.

The potential resulting from the rotation, however, is still too general to allow for a complete investigation of the domain of integrability. So in the case of degree four potentials, we will not insist on the exhaustive nature of our analysis, contrary to what was done in Sec. III. In what follows we will limit ourselves to potentials which are parity symmetric separately in x and y, of the form

$$V = y^4 + ay^2x^2 + bx^4. \tag{4.3}$$

Before proceeding further, we remark that, in this form, the potential possesses two-well known integrable cases.[8] The first one is the usual separable case:

$$V = y^4 + bx^4, \tag{4.4}$$

which possesses an integral quadratic in the velocities. [In the special case $b = 1$, the separable potential is still of the form (4.3) after a rotation of $\pi/4$, i.e., $V = x^4 + 6x^2y^2 + y^4$.] The second case is the rotationally invariant potential:

$$V = (y^2 + x^2)^2, \tag{4.5}$$

whose second integral of motion is just the angular momentum. Actually, this potential is a member of the Darboux family we introduced in Ref. 3.

A further case of quartic potential is encountered within the new family of integrable potentials presented in Ref. 3, and which possess integrals of motion quadratic in velocities:

$$V = y^4 + \tfrac{3}{4}x^2y^2 + \tfrac{1}{16}x^4. \tag{4.6}$$

A. Painlevé analysis of the equations of motion

The equations of motion are

$$\ddot{y} = -4y^3 - 2ax^2y, \tag{4.7}$$

$$\ddot{x} = -2axy^2 - 4bx^3. \tag{4.8}$$

There are three kinds of singularities. In the first case, both x and y diverge as $(t - t_0)^{-1}$; in the second case, y diverges as $(t - t_0)^{-1}$, and x does not diverge that fast while the reverse is true in the third case.

Let us first consider the second case. Equation (4.7) fixes the coefficient of the leading term of y. In analogy with Sec. IIIA, if we define γ by

$$y = \gamma(t - t_0)^{-1} + \epsilon(t - t_0)^{-1},$$

with $\epsilon \to 0$ as $t \to t_0$, we find

$$\gamma^2 = -\tfrac{1}{2}.$$

The possible behaviors for x are determined by (4.8). Let again s be the power dependence of x. We find

$$s(s - 1) = a. \tag{4.9}$$

Here, since the equation is even separately in x and y, it would suffice to satisfy the Painlevé property for x^2 and y^2. A necessary condition for this is that s be an integer or a half-integer. A second necessary condition can be found by considering the third case of singularity described above. By direct analogy, one finds

$$a/b = u(u - 1), \tag{4.10}$$

with u an integer or a half-integer. Let us now consider the singularity where both x and y diverge as $(t - t_0)^{-1}$, i.e.,

$$x = \alpha(t - t_0)^{-1} + \epsilon_1(t - t_0)^{-1},$$

$$y = \beta(t - t_0)^{-1} + \epsilon_2(t - t_0)^{-1},$$

where ϵ_1 and ϵ_2 go to zero as $t \to t_0$, and α and β satisfy

$$2 = -4\beta^2 - 2a\alpha^2, \tag{4.11}$$

$$2 = -2a\beta^2 - 4b\alpha^2. \tag{4.12}$$

This system has in general several couples of solution (α,β). For each of them a resonance will occur whenever the determinant of the matrix M vanishes, where M is given by

$$M = \begin{pmatrix} (n-1)(n-2) + 12\beta^2 + 2a\alpha^2 & 4a\alpha\beta \\ 4a\alpha\beta & (n-1)(n-2) + 2a\beta^2 + 12b\alpha^2 \end{pmatrix}.$$

Again, n only enters through the expression $N = (n - 1)(n - 2)$, and $N = 6$ is always a solution corresponding to $n = -1$ and $n = 4$. Indeed, using (4.11) and (4.12), the matrix M becomes, with $N = 6$,

$$M = \begin{pmatrix} -4a\alpha^2 & 4a\alpha\beta \\ 4a\alpha\beta & -4a\beta^2 \end{pmatrix}.$$

The other solution N' for N satisfies

$$N' + 6 = -(12 + 2a)\beta^2 - (2a + 12b)\alpha^2. \tag{4.13}$$

A necessary condition for the Painlevé property is that N' be of the form:

$$N' = v(v - 1), \tag{4.14}$$

for every choice of the couple (α,β) that solves (4.11) and (4.12), where v is an integer that could very well depend on the couple. However, a singularity of the second or third kind with s or u half-integers, in a rotated frame, appear as a singularity of the kind considered now with v a half-integer. In order to recover them we will also accept half-integer values of v.

The problem of finding all the possible values of a and b that satisfy (4.9), (4.10), and (4.14) with N' given by (4.13) for every choice of a couple (α,β) that solves (4.11) and (4.12) is

still very complicated. In analogy with what was done in Ref. 3, we will try to find some solutions with the help of a simplifying assumption which is a slight extension of Eq. (25) of Ref. 3. This assumption is that the quantity

$$(12 + 2a)\beta^2 + (2a + 12b)\alpha^2,$$

which appears on the right-hand side of equation (4.13), is a linear combination, *as a polynomial in α and β*, of the quantities $4\beta^2 + 2a\alpha^2$ and $2a\beta^2 + 4b\alpha^2$ that appear in Eqs. (4.11) and (4.12), with coefficients λ and μ, which as a consequence satisfy

$$12 + 2a = 4\lambda + 2a\mu, \tag{4.15}$$

$$2a + 12b = 2a\lambda + 4b\mu. \tag{4.16}$$

Then, for any α and β, and in particular any solution of the system (4.11) and (4.12), the following will be automatically satisfied:

$$N' + 6 = 2\lambda + 2\mu.$$

The problem is now to choose a and b such that

$$2\lambda + 2\mu - 6 = v(v-1). \tag{4.17}$$

In that way, we will obtain only systems for which the resonances are the same for every choice of the couple (α,β), which is certainly not a necessary condition. Thus we don't expect this search to be exhaustive. Even the arithmetic system (4.9), (4.10), (4.14) through (4.11), (4.12), is too vast to be solved in all generality. Here are the solutions we have found:

(a) $s = \frac{3}{2}$, $u = 4$, $v = 4$, $\lambda = 0$, $\mu = 9$,

$$V = y^4 + \tfrac{3}{4}y^2x^2 + x^4/16. \tag{4.18}$$

This polynomial has been described in Ref. 3. It has a second integral quadratic in the velocities.

(b) $s = 2$, $u = 2$, $v = 2$, $\lambda + \mu = 4$.

$$V = (y^2 + x^2)^2. \tag{4.19}$$

This polynomial has also been described in Ref. 3 (for $\lambda = 0$, $\mu = 4$). It is obviously integrable.

(c) $s = \frac{3}{2}$, $u = 3$, $v = 7$, $\lambda = -9$, $\mu = 33$,

$$V = y^4 + \tfrac{3}{4}y^2x^2 + x^4/8. \tag{4.20}$$

This polynomial was not known to be integrable so far. As will be shown in the next subsection, it indeed possesses a second integral quartic in the velocities.

(d) $s = \frac{3}{2}$, $u = 17$, $v = 7/2$, $\lambda = 39/40$, $\mu = 32/5$,

$$V = y^4 + \tfrac{3}{4}y^2x^2 + 3x^4/1088. \tag{4.21}$$

Here v is a half-integer but so is s. No rotation of x and y can be found that would completely separate integers from half-integers for all singularities at the same time. Thus, there is no change of variable for which this system would be Painlevé. However, it could be "weak Painlevé" in the sense of Ref. 3.

B. Direct search for the integrals of motion

The first compatibility condition (2.10) applied on a potential of the form (4.3) gives

$$(a-2)f_3 + (2b-a)f_1 = 0. \tag{4.22}$$

We first look for solutions with the f_i's constant; f_4 can be taken equal to zero by adding to C the suitable multiple of the square of the Hamiltonian.

Provided f_i, a, and b obey (4.22), the equations for the g_i can be integrated to give

$$\begin{aligned} g_0 &= f_0(4ax^2y^2 + 4bx^4) + f_1(\tfrac{2}{3}ax^3y + 4xy^3) + f_2(2 - a/3)y^4, \\ g_1 &= f_0(-\tfrac{8}{3}ayx^3) + f_1[(3a-6)x^2y^2 + (3 - a/6)x^4] \\ &\quad + f_2(\tfrac{4}{3}axy^3 + \tfrac{4}{3}ayx^3) + f_3(3 - a/6)y^4, \\ g_2 &= f_0(\tfrac{2}{3}ax^4) + f_1(2 - a/3)x^4 + f_3(\tfrac{2}{3}axy^3 + 4bx^3y). \end{aligned} \tag{4.23}$$

Finally, the second compatibility relation is used to obtain a system of equations for the f_i's, a and b, the solution of which gives integrable quartic potentials:

$$\begin{aligned} &f_1(a-6b)(a-42b) = 0, \\ &a(a-6b)(2f_0 - f_2) = 0, \\ &2af_3 + f_1W_1(a,b) = 0, \\ &(\tfrac{40}{3})(a-6)f_2 + f_1W_2(a,b) = 0, \\ &a(1 - 4a/3)f_0 + f_1W_3(a,b) = 0, \\ &f_3(-a^2/3 + 16a - 48) + f_1W_4(a,b) = 0, \\ &f_3(132 - 32a + 29a^2/3) + f_1W_5(a,b) = 0, \end{aligned} \tag{4.24}$$

where the W_i's are rather complicated expressions in term of a and b.

The solutions of system (4.24) is quite analogous to that of the system (3.18). One finds immediately that the system has a solution for

$$f_0 = 1, \quad f_i = 0, \quad i = 2,3,4.$$

The seven Eqs. (4.24) with the condition (4.22) reduce then to

$$a(1 - 4a/3) = 0, \quad a(a-6b) = 0. \tag{4.25}$$

—The case $a = 0$ corresponds to the separable potential (4.4), the constant of motion being the square of a quadratic constant of motion.

—The case $a = 6b$, $a = \frac{3}{4}$, i.e., $a = \frac{3}{4}$, $b = \frac{1}{8}$,

$$V = y^4 + \tfrac{3}{4}x^2y^2 + \tfrac{1}{8}x^4 \tag{4.26}$$

is just the new potential (4.20) provided by the Painlevé analysis. The constant of motion associated with it is written as

$$\begin{aligned} C = {} & \dot{x}^4 + (24x^2y^2 + 4x^4)\dot{x}^2 - 16x^3y\dot{x}\dot{y} + 4x^4\dot{y}^2 \\ & + 4x^8 + 16x^6y^2 + 16x^4y^4. \end{aligned} \tag{4.27}$$

However, no trace was found of a solution associated with the potential (4.21) at this order, even with the general form of the f_i's. This motivated an investigation of the existence of a second integral of motion at order five or six in the velocities. We will not present here any of these cumbersome calculations. Their result is that the potential in question does not possess a second integral of motion up to order six. This, in itself, does not exclude the existence of higher order integrals. However, numerical studies of the surfaces of section exhibit a completely chaotic behavior, which is the signature of nonintegrability.

V. PRECISE DEFINITION OF THE WEAK PAINLEVÉ PROPERTY

As we have seen, the nonintegrable potential (4.21) does not have the usual Painlevé property. No rotation of x and y

completely separates integer powers from half-integer ones for all singularities at the same time, which would be necessary for the system to be Painlevé in terms of the rotated x^2 and y^2. However, according to the definition of Ref. 3, which was only grossly stated at that stage, this potential appears to have the weak-Painlevé property. Indeed, we said that for an integrable polynomial potential of degree $p+2$ the solutions should be expansions in terms of $(t-t_0)^{1/r}$, and we expected there r to be equal to p. The leading behavior in the neigborhood of a singularity is as $(t-t_0)^{-2/p}$, and one of the resonances is always at $2+4/p$ which makes p a reasonable candidate for the denominator of the natural power. Note, however, that, while, for p odd, there is no other candidate than p, for even p, $p/2$ is just as natural as p itself. In the case of the potential (4.21), p is equal to 2, and we do have an expansion in terms of $(t-t_0)^{1/2}$. On the other hand, for all known integrable polynomials of even degree, the expansions are really in terms of $(t-t_0)^{2/p}$. This is actually the case of the polynomials of the Darboux class[8] and also for the new class described in Ref. 3 (in terms of x^2 and y rather than x and y). Finally it is also true for the potential (4.20). Thus it appears that, for even p, whenever an expansion is found in powers of $(t-t_0)^{1/(p/2)}$, the potential is integrable. On the other hand, the case (4.21) above, which is associated with an expansion in $(t-t_0)^{1/p}$, is not integrable.

This has motivated a further investigation. We perturb potential (4.20) through the addition of a term that introduces half-integer powers. The new potential is

$$V=y^4+\tfrac{3}{4}y^2x^2+\tfrac{1}{8}x^4+\lambda x. \tag{5.1}$$

The addition of the linear term excludes the existence of a second constant of the motion quartic in the velocities. Indeed it is not possible to add to the constant (4.27) terms dependent on λ so as to recover a new constant for nonvanishing λ. This does not entirely preclude the existence of a higher order integral of the motion. None was found, however up to order six. Again, the nonintegrability of this system has been confirmed by numerical studies of the surface of section which exhibit large scale chaos.

As for the Painlevé property, the resonances are, of course, not modified by the addition of a nondominant term. One can check that the new term does *not* introduce any logarithms at the resonances. The only modification is thus the introduction of half-integer powers. So we have here a second case where nonintegrability is associated with an expansion in terms of $(t-t_0)^{1/2}$ with $p=2$.

We can thus conclude with the following precise definition of the weak Painlevé property. We demand an expansion in terms of rational powers of $(t-t_0)$. The denominator of the rational exponent should be

p for odd p's,

$p/2$ for even p's,

where the degree of the polynomial potential is $p+2$. Note that this choice is rather more natural than always taking p as denominator of the natural power because this is just the denominator of the exponent of the leading behavior. Indeed the leading behavior being $(t-t_0)^{-2/p}$, the denominator is p when p is odd but is $p/2$ when p is even.

VI. CONCLUSION

In this work we have presented some new results based on the complex-plane singularity analysis of the equations of motion, using the weak Painlevé concept we introduced in Ref. 1. The dynamical systems considered correspond to the motion of a particle in a two-dimensional homogeneous polynomial potential of degree three and four. The latter limitation was imposed by the sheer complexity of the problem. Two methods have been used in parallel in order to investigate the integrability, Painlevé analysis, and direct computation of the integral. For the case of cubic interactions we have reduced the condition for the validity of the Painlevé property to a simple arithmetical equation of which we were able to exhibit all the solutions, leading to novel cases. For the case of quartic Hamiltonians, our search was not exhaustive. It led, nevertheless, to the discovery of new Painlevé potentials. The integrability was, in each case, demonstrated explicitly through the direct calculation of the integrals of motion. This search is performed systematically order by order. It cannot thus constitute a criterion of nonintegrability whenever it fails to give a positive result. However, the nonintegrability up to integrals of order six for a quartic potential which seemingly satisfies the weak Painlevé property, combined with the "nonusual" way this property is satisfied, makes the latter a serious candidate for nonintegrability. This has led us into considering in detail the notion of the "natural power," which is basic to the weak Painlevé concept.

So what emerges as a conclusion, from the results we presented in this series of papers, is that the Painlevé property is a most useful tool for the investigation of the integrability of dynamical systems, which is far from having shown the limits of its possibilities. Actually the implementation of this property to dissipative two-dimensional systems appears to be within reach.

[1] A. Ramani, B. Dorizzi, and B. Grammaticos, Phys. Rev. Lett. **49**, 1539 (1982).
[2] M. J. Ablowitz, A. Ramani, and H. Segur, Lett. Nuovo Cimento **23**, 333 (1978); M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. **21**, 715 (1980).
[3] B. Dorizzi, B. Grammaticos, and A. Ramani, J. Math. Phys. **24**, 2282 (1983).
[4] J. Bertrand, J. Math. (i) **XVII**, 121 (1852).
[5] M. Hénon, Phys. Rev. B **9**, 1921 (1974).
[6] B. Dorizzi, B. Grammaticos, V. Papageorgiou, and R. Padjen, preprint, 1982.
[7] M. Hénon and C. Heiles, Astron. J. **69**, 73 (1964).
[8] T. Bountis and H. Segur, in *The Proceedings of the Workshop on "Mathematical Systems,"* edited by M. Tabor and Y. Trève, AIP Conference Proceedings, Vol. 88 (AIP, New York, 1982).
[9] Y. F. Chang, M. Tabor, and J. Weiss, J. Math. Phys. **23**, 531 (1982).
[10] L. J. Hall, Lawrence Livermore Laboratory Preprint UCID-18980 Rev. 1, 1982, submitted to Ann. Phys (N.Y.).
[11] B. Grammaticos, B. Dorizzi, and R. Padjen, Phys. Lett. A **83**, 111 (1982).
[12] J. Greene, quoted in Ref. 9.

A deformation of the general zero-curvature equations associated to simple Lie algebras

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We construct deformations and rational reductions for all the general zero-curvature equations associated to simple complex Lie algebras [known as AKNS equations for sl(2, $\mathbb{C}$)].

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I. INTRODUCTION

One of the fundamental properties of the Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx}, \quad u = u(x,t), \tag{1.1}$$

is that it has an infinite number of local conservation laws: that is, there is an infinite sequence of identities

$$\partial_t H_q = \partial J_q, \quad q = 1,2,... \tag{1.2}$$

that follow formally from (1.1). (We write $\partial_t = \partial/\partial t$, $\partial \equiv \partial/\partial x$.) The "conserved densities" H_q and "fluxes" J_q are elements of $\mathbb{C}[u^{(j)}]$ = the set of differential polynomials in u, i.e., polynomials in u and its x-derivatives $u^{(j)} = \partial^j(u)$.

Here is the charming construction, due to Gardner,[1] of these conservation laws. Consider another equation

$$w_t = 6ww_x - w_{xxx} + 6\epsilon^2 w^2 w_x, \quad w = w(x,t). \tag{1.3}$$

It is easy to check that if w satisfies (1.3), then $u(x,t)$, given by the formula

$$u = w + \epsilon^2 w^2 + \epsilon w_x \tag{1.4}$$

satisfies (1.1). Now, rewrite (1.3) in the conservation form

$$\partial_t w = \partial(3w^2 - w_{xx} + 2\epsilon^2 w^3), \tag{1.5}$$

invert (1.4) (understood as an automorphism of differential rings $\mathbb{C}[u^{(j)}]\,[[\epsilon]] \to \mathbb{C}[w^{(j)}]\,[[\epsilon]]$)

$$w = u + \sum_{k=1}^{\infty} \epsilon^k P_k, \quad P_k \in \mathbb{C}[u^j], \tag{1.6}$$

substitute (1.6) into (1.5), and identify ϵ^q-coefficients on both sides of the resulting equality: you get (1.2).

What is the meaning of the Gardner trick? Notice that under the homomorphism (1.4), conservation laws (1.2) for the KdV equation (1.1) become conservation laws for the Gardner equation (1.3) which, therefore, is also an *integrable* system, that is, it has an infinite number of conservation laws. Thus, starting with the KdV equation (1.1), we have a curve (1.3) parametrized by ϵ in the space of evolution equations, and an *integrable curve* at that. Moreover, the map (1.4) tells us that we also have a *reduction* of our curve, that is, a regular map which sends any point on the curve into a base point with parameter $\epsilon = 0$ and is the identity at $\epsilon = 0$ (We do not distinguish between evolution equations and their solutions; see, e.g., Ref. 2 for the spirit of algebraic treatment of evolution equations.) It is, then, irresistible to conjecture that (a) most, if not all, integrable systems currently in circulation (see, e.g., Refs. 3–5) can be included in integrable one-parameter families, which we call *deformations*, and, moreover, (b) these curves carry with them reductions. There is some evidence available to back up this conjecture (see, e.g., Refs. 6 and 7), though it is not at all clear what could be the underlying reasons for the existence of such a general phenomenon.

The main result of this paper establishes the existence of deformations and *rational* reductions for all the general zero-curvature equations associated to simple Lie algebras.[5] Details will be given in the course of the paper via the following route: in Sec. II, we review the general zero-curvature equations associated to simple complex Lie algebras; in Sec. III, we study two different coordinate systems on the tangent bundle of the manifold of Cartan subalgebras of a given simple Lie algebra. We find that these coordinate systems are related by a rational map. In Sec. IV, we interpret constructions of Sec. III as providing deformations and desired reductions.

II. THE GENERAL ZERO-CURVATURE EQUATIONS

In this section, we summarize the Wilson construction of the general zero-curvature equations.[5]

Let $\mathfrak{g}$ be a complex simple Lie algebra, F a regular semisimple element of $\mathfrak{g}$, $\mathcal{f}$ a unique Cartan subalgebra containing F, so that we have

$$\mathfrak{g} = \mathcal{f} \oplus [\mathcal{f}, \mathfrak{g}] = \mathcal{f} \oplus \operatorname{Im} \operatorname{ad} F. \tag{2.1}$$

Let $l = \dim \mathcal{f}$ denote the rank of $\mathfrak{g}$ and let $R \subset \mathcal{f}^*$ be the set of roots of $(\mathfrak{g}, \mathcal{f})$. For every $\alpha \in R$, let E_α be a nonzero element of the corresponding root space, so that

$$[\mathcal{f}, \mathfrak{g}] = \bigoplus_{\alpha \in R} \mathbb{C} E_\alpha. \tag{2.2}$$

Let u_α, $\alpha \in R$, be differentially independent variables, and let $B = \mathbb{C}[u_\alpha^{(j)}]$ be the differential algebra of polynomials in variables $u_\alpha^{(j)}$ with the derivation ∂ acting on B through $\partial(u_\alpha^{(j)}) = u_\alpha^{(j+1)}$. We introduce a grading on B by setting $\deg u_\alpha^{(j)} = j + 1$. We set $\tilde{\mathfrak{g}} = B \underset{\mathbb{C}}{\otimes} \mathfrak{g}$ and extend the derivation ∂ and the grading deg to $\tilde{\mathfrak{g}}$ by $\partial(1 \otimes \mathfrak{g}) = 0$, $\deg(1 \otimes \mathfrak{g}) = 0$.

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Define $u = \Sigma\, u_\alpha \otimes E_\alpha$, which, for brevity, we shall write as $\Sigma\, u_\alpha E_\alpha$. Let λ be a formal parameter commuting with everything; set

$$U = -u + \lambda F \in \tilde{\mathfrak{g}}. \tag{2.3}$$

For $V \in \tilde{\mathfrak{g}}\,[\lambda\,]$, our general zero-curvature equations[5]

$$[\partial - U, \partial_t - V\,] = 0 \tag{2.4}$$

are equivalent to

$$\partial_t u = -\partial V + [\,U,V\,]. \tag{2.4'}$$

These equations make sense if and only if the rhs of (2.4') does not depend upon λ and lies in $[\widetilde{\mathscr{f}, \mathfrak{g}}] = B \underset{\mathbb{C}}{\otimes} [\mathscr{f}, \mathfrak{g}]$. Here is the full description of all possible V's:

Fix a natural number $r \geqslant 1$ and an element $v \in \mathscr{f}$. Let

$$W = \sum_{i=0}^{\infty} v_i \lambda^{-i}, \quad v_i \in \tilde{\mathfrak{g}}, \quad v_0 = v \tag{2.5}$$

be a unique solution of the equation $\partial W = [U,W]$ such that v_i is homogeneous of degree i. Set

$$V = \sum_{i=0}^{r} v_i \lambda^{r-i}, \quad V_- = \sum_{i=r+1}^{\infty} v_i \lambda^{r-i}. \tag{2.6}$$

From $\lambda^r W = V + V_-$, we have

$$-\partial V + [\,U,V\,] = \partial V_- - [\,U,V_-\,], \tag{2.7}$$

and since the lhs does not involve any negative powers of λ whereas the rhs does not contain any positive powers, (2.7) is λ-independent. Picking out the terms of order zero in λ, we rewrite (2.4') as

$$u_t = -v_{r,x} + [\,u,v_r\,] = [\,F, -v_{r+1}\,], \tag{2.8}$$

which shows that the rhs belongs to $[\widetilde{\mathscr{f},\mathfrak{g}}] = \widetilde{\operatorname{Im}\operatorname{ad} F}$.

Now denote by $\partial_t = \partial_t(v,r)$ the evolutionary (i.e., commuting with ∂) derivation of B, which is defined by (2.8) via

$$\partial_t(v,r)u_\alpha = \alpha\text{-component of } [\,F, -v_{r+1}\,] \text{ in } [\widetilde{\mathscr{f},\mathfrak{g}}]. \tag{2.9}$$

These are the equations we are going to deform. The properties of these equations are given in the following proposition:

Proposition 2.1: (i) If $v \neq 0$ then $\partial_t(v,r) \neq 0$. Thus we get l linearly independent derivations for each r. (ii) Let K denote the Killing form on $\mathfrak{g}$ naturally extended to $\tilde{\mathfrak{g}}$, and set $H_s = H_s(v) = s^{-1}K(v_{s+1}, F)$. Then the elements H_s, $s \geqslant 1$ are common nontrivial (i.e., they do not lie in ∂B) conserved densities of all evolution equations (2.9) (that is, $\partial_t H_s \in \partial B$). Thus we obtain l linearly independent conservation laws for each $s \geqslant 1$. (iii) Equations (2.9) can be written in Hamiltonian form

$$\partial_t(v,r)u_\alpha = -\alpha(F)\frac{\delta H_{r+1}}{\delta u_{-\alpha}}, \quad \alpha \in R, \tag{2.10}$$

where $\delta/\delta u_\alpha$ is the functional derivative with respect to u_α. (See, e.g., Ref. 2 for the differential-algebraic version of calculus.) Thus all derivations $\partial_t(v,r)$ (or corresponding "flows") commute with each other.

The proofs follow from the general theory of Lax[4] equations. We shall not need them: our object of study is just Eqs. (2.4).

III. MANIFOLD OF CARTAN SUBALGEBRAS

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\mathscr{H}$ the set of all Cartan subalgebras in $\mathfrak{g}$. $\mathscr{H}$ is a homogeneous space for the Lie group Aut $(\mathfrak{g})$ and we may regard $\mathscr{H}$ as a nonsingular algebraic variety. In this section, we study relations between two coordinate systems on (open piece of) $\mathscr{H}$. (To avoid any confusion with the notations of the preceding section, the reader would do well to ignore temporarily the existence of Sec. II).

Let F be a semisimple element of $\mathfrak{g}$ such that the dimension of the centralizer $\mathfrak{g}^F$ of F in $\mathfrak{g}$ is minimal; in other words, F is regular semisimple. Then $\mathscr{f} := \mathfrak{g}^F$ is a Cartan subalgebra of $\mathfrak{g}$, and the rank of $\mathfrak{g}$ is given by rk $\mathfrak{g} := \dim \mathscr{f}$. We then have

$$\mathfrak{g} = \mathscr{f} \oplus [F,\mathfrak{g}], \quad [\mathscr{f}, \mathfrak{g}] = [F,\mathfrak{g}] = \operatorname{Im}(\operatorname{ad} F).$$

Let $\mathscr{H}'$ be the open subset of $\mathscr{H}$ consisting of all $\mathscr{f}' \in \mathscr{H}$ satisfying the transversality condition $\mathscr{f}' \cap [F, \mathfrak{g}] = \{0\}$. We define a smooth map $P: \mathscr{H}' \to [F, \mathfrak{g}]$ by requiring $(F - P(\mathscr{f}')) \in \mathscr{f}'$ for all $\mathscr{f}' \in \mathscr{H}'$. Then $P(\mathscr{f}')$ uniquely determines $\mathscr{f}'$ by $\mathscr{f}' = \mathfrak{g}^{F - P(\mathscr{f}')}$ provided $F - P(\mathscr{f}')$ happens to be regular semisimple; we denote by $\mathscr{H}''$ the set of all such $\mathscr{f}' \in \mathscr{H}'$ (for which it does happen). Then $\mathscr{f} \in \mathscr{H}''$ [since $F - P(\mathscr{f}) = F$] and $\mathscr{H}''$ is open in $\mathscr{H}$ (since $\mathscr{H}'' = [\mathscr{H}'$ which is open in $\mathscr{H}]$ $\cap$ [{the set of all regular semisimple elements} which is open open in $\mathfrak{g}$]). Therefore P is smooth and injective on the nonempty open subset $\mathscr{H}''$ of $\mathscr{H}$. Since $\dim \mathscr{H} = \dim [F,\mathfrak{g}]$ $(= \dim \operatorname{Im} P)$, it follows that the differential dP of the map P is surjective at all points $\mathscr{f}' \in \mathscr{H}''$.

Now let us consider the tangent bundle $\mathrm{T}(\mathscr{H}'')$ of $\mathscr{H}''$. Regarding P as providing a coordinate system for $\mathscr{H}''$, we identify $\mathrm{T}(\mathscr{H}'')$ with $\mathrm{T}(P(\mathscr{H}'')) \cong P(\mathscr{H}'') \times [F, \mathfrak{g}]$, the tangent bundle of $P(\mathscr{H}'')$. Denote the coordinate system thus obtained on $\mathrm{T}(\mathscr{H}'')$ by $(P, P_x) \in P(\mathscr{H}'') \times [F,\mathfrak{g}]$.

Another coordinate system (P,u) on $P(\mathscr{H}'') \times [F,\mathfrak{g}]$ may be defined as follows. Any $w \in \mathfrak{g}$ determines a holomorphic vector field X_w on $\mathscr{H}$ as the generator of a one-parameter family of diffeomorphisms $\exp(\tau \operatorname{ad} w)$ restricted to $\mathscr{H}$: for any $f \in C^\infty(\mathscr{H})$,

$$(X_w f)(\mathscr{f}') = \lim_{\tau \to 0} \tau^{-1}[f((\exp(\operatorname{ad} \tau w))\cdot \mathscr{f}') - f(\mathscr{f}')].$$

The map $w \mapsto X_w$ is a Lie algebra homomorphism. Denote by $X_w(\mathscr{f}') \in T_{\mathscr{f}'}(\mathscr{H})$ the value of X_w at $\mathscr{f}'$. Then $X_w(\mathscr{f}') = 0$ iff $w \in \mathscr{f}'$. Thus the correspondence

$$(\mathscr{f}', X_{F-u}(\mathscr{f}')) \leftrightarrow (P(\mathscr{f}'), u), \quad u \in [F, \mathfrak{g}]$$

defines our second coordinate system on $\mathrm{T}(\mathscr{H}'')$.

Our goal is to connect these coordinate systems. To obtain a connection between (P,P_x) and (P,u), consider a holomorphic curve $\gamma: \overline{B} \to \mathscr{H}''$ defined on an open ball $\overline{B} \subset \mathbb{C}$ containing zero. Put $P(\tau) = P(\gamma(\tau))$ for $\tau \in \overline{B}$, set $P := P(0)$, $P_x = dP(\tau)/d\tau|_{\tau=0}$, and let (P,u) correspond to (P,P_x). For the Cartan subalgebra $\gamma(\tau)$, when $\tau \to 0$, its general regular semisimple element near $F - P$ [recall that $\mathfrak{g}^{F-P} = \gamma(0)$] can be written in two different forms (according to the two coordinate systems introduced above) as

$$F - P - \tau P_x + O(\tau^2),$$

and

$$F-P+\tau[F-u,F-P]+O(\tau^2)$$
$$=(\exp \operatorname{ad} \tau(F-u))\cdot(F-P)+O(\tau^2).$$

Since $\gamma(\tau)$ is commutative, we have

$$[F-P-\tau P_x, F-P+\tau[F-u,F-P]]=O(\tau^2),$$

or, putting

$$T:=\operatorname{ad}(F-P), \tag{3.1}$$

we get

$$T^2(P-u)=T(P_x). \tag{3.2}$$

We shall now analyze the correspondence $(P,P_x)\leftrightarrow(P,u)$ given by (3.2), and we regard F as variable as well (we shall need this later). We note that the results below require only the assumption that $\mathfrak{g}$ is a simple Lie algebra over a field k of characteristic zero, and in such a context we formulate our statements (one often needs $k=\mathbb{R}$ rather than $k=\mathbb{C}$. We want to cover this case as well).

The following lemma from linear algebra provides the basis for our analysis.

Lemma 3.1: Let L be a vector space over a field k, with fixed basis $w_1,\ldots,w_n$. Let r be an integer with $1\leqslant r\leqslant n$, and consider the algebraic variety

$$M=\{(T,T')\in \operatorname{End} L\times \operatorname{End} L \mid \operatorname{rank} T'\leqslant r\}.$$

Define an element $A\in k[M]$ (= regular functions on M) by

$$A(T,T')w_1\wedge\cdots\wedge w_n$$
$$=TT'w_1\wedge\cdots\wedge TT'w_r\wedge w_{r+1}\wedge\cdots\wedge w_n. \tag{3.3}$$

(Both sides lie in the one-dimensional space $\Lambda^n(L)$ with the basic vector $w_1\wedge\cdots\wedge w_n$.) Then there exists $T''\in k[M]\otimes_k \operatorname{End} L$ such that

$$T'TT''=T'A \tag{3.4}$$

in $k[M]\otimes_k \operatorname{End} L$ [in other words, considered as a regular function on M with values in End L, T'' satisfies $T'TT''(T,T')=T'A(T,T')$].

Proof: Write $TT'w_i=\sum_{j=1}^n a_{ij}w_j$, $1\leqslant i\leqslant n$, where $a_{ij}\in k[M]$. Define the matrix $a=(a_{ij})_{1\leqslant i,j\leqslant r}$, then $A=\det a$. Next we take the matrix $a'=(a'_{ij})_{1\leqslant i,j\leqslant r}$ such that $aa'=a'a=A\mathbb{1}_r$, so that $a'_{ij}\in k[M]$. Set, for $1\leqslant i\leqslant r$, $\bar{w}_i=\sum_{j=1}^r a'_{ij}w_j$, so $Aw_i=\sum_{j=1}^r a_{ij}\bar{w}_j$. Now we can define T'' (for the fixed T and T') by $T''w_j=T'\bar{w}_j$, $1\leqslant j\leqslant r$, $T''w_j=0$, $r<j\leqslant n$. Since, for $1\leqslant i\leqslant r$, $TT'\bar{w}_i=TT'\sum_{j=1}^r a'_{ij}w_j=\sum_{j=1}^r a'_{ij}TT'w_j=\sum_{j=1}^r a'_{ij}\sum_{k=1}^n a_{jk}w_k$ $\equiv\sum_{j,k=1}^r a'_{ij}a_{jk}w_k$ (mod $L_r:=kw_{r+1}+\cdots+kw_n$) $=Aw_i$ (mod L_r), we have, again for $1\leqslant i\leqslant r$, $T''TT'\bar{w}_i=T''(Aw_i \bmod L_r)=T''(Aw_i)=AT''w_i=AT'\bar{w}_i$. Thus $(T''TT'-AT')=0$ on $\bar{L}^r:=k\bar{w}_1+\cdots+k\bar{w}_r$ and since $Aw_i\in\bar{L}^r$, we get $(T''TT'-AT')A=0$ on $L^r:=kw_1+\cdots+kw_r$. Now rk $T'\leqslant r$, so $T'(L)=T'(L^r)$. Thus if $A\neq 0$, then $T''TT'-AT'=0$ on L. But M is irreducible, and A is regular and not identically zero on M. Therefore $T''TT'-AT'\equiv 0$ always.

Q.E.D.

We now use Lemma 3.1 in the situation: $L=\mathfrak{g}$, $r=\dim\mathfrak{g}-\operatorname{rk}\mathfrak{g}$, and T, T' on M are given by $T=\operatorname{ad}(F-P)$, $T'=\operatorname{ad}F$ where $P\in\mathfrak{g}$ and $F\in\mathfrak{g}$ is regular semisimple. If F is regular semisimple, i.e., $\dim\mathfrak{g}^F=\operatorname{rk}\mathfrak{g}$, and $[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}$, we choose a basis $w_1,\ldots,w_n$ in $\mathfrak{g}$ in the following manner: take $w_1,\ldots,w_r$ such that $[F,w_1]$, $\ldots,[F,w_r]$ are linearly independent (we can do this since F is regular so $\dim(\operatorname{Im}\operatorname{ad}F)=n-\dim\mathfrak{g}^F=r$), then $TT'w_1=T([F,w_1]),\ldots,TT'w_r=T([F,w_r])$ are also linearly independent (since $[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}$). Now choose complementary vectors $w_{r+1},\ldots,w_n$ such that $A\neq 0$, see (3.3). Now define, via Lemma 3.1, the element

$$S=A^{-1}T''T. \tag{3.5}$$

Claim: S is the projection of $\mathfrak{g}$ onto $[F,\mathfrak{g}]$ along $\mathfrak{g}^{F-P}$. Indeed, $ST'=A^{-1}T''TT'=A^{-1}AT'=T'$, thus $S=\mathrm{Id}$ on $\operatorname{Im}T'=\operatorname{Im}\operatorname{ad}F$. On the other hand, if $y\in\mathfrak{g}^{F-P}=\operatorname{Ker}\operatorname{ad}(F-P)=\operatorname{Ker}T$, that is, $Ty=0$, then $Sy=A^{-1}T''Ty=0$. To sum up, S is a rational function of $(F,P)\in\mathfrak{g}\times\mathfrak{g}$ defined whenever $\dim\mathfrak{g}^F=\operatorname{rk}\mathfrak{g}$ and $[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}$.

Applying this claim to our basic equation (3.2), written in the form

$$P_x=S([F-P,P-u]), \tag{3.6}$$

we get P_x as a regular function on the following Zariski-locally-closed subset Z of $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$:

$$Z:=\{(F,P,u)\mid \dim\mathfrak{g}^F=\operatorname{rk}\mathfrak{g};\ P,u\in[F,\mathfrak{g}];$$
$$[F,\mathfrak{g}]\cap\mathfrak{g}^{F-P}=\{0\}\}. \tag{3.7}$$

Denote by Y the analogous set of (F,P,P_x) obtained by replacing u by P_x in (3.7).

Now let Q be the projection of $\mathfrak{g}$ onto $[F-P,\mathfrak{g}]$ along $\mathfrak{g}^{F-P}$. Then, as above for S, Q is a rational function of $(F,P)\in\mathfrak{g}\times\mathfrak{g}$ defined whenever $F-P$ is regular semisimple (perhaps we should stress that in all matters unrelated to our problem of connecting coordinate systems, we treat F and P as free variables).

Turning back to our basic equation (3.2), we rewrite it with the help of Q as

$$T(P-u)=Q(P_x). \tag{3.8}$$

Since $\operatorname{ad}F$ is an isomorphism on $[F,\mathfrak{g}]=\operatorname{Im}(\operatorname{ad}F)$, we can define P', $u'\in[F,\mathfrak{g}]$ by $P=[F,P']$, $u=[F,u']$. Then $P-u=T'(P'-u')$ and we have, by Lemma 3.1,

$$A(P-u)=AT'(P'-u')=T''TT'(P'-u')$$
$$=T''T(P-u)=T''(Q(P_x)).$$

This shows that u is a regular function on the Zariski-locally-closed set

$$Y':=\{(F,P,P_x)\in Y\mid (F-P) \text{ is regular semisimple}\}$$

(recall that $A\neq 0$ on Y). Denoting $Z'=\{(F,P,u)\in Z\mid (F-P$ is regular semisimple$\}$, we collect the results of this reasoning in the following theorem.

Theorem 3.2: The maps $Z\to Y$, given by (3.6), and $Y'\to Z'$, given by (3.8), are regular. In particular, our basic Eq. (3.2) determines an everywhere-defined birational correspondence $Y'\leftrightarrow Z'$.

Now we can analyze different asymptotics connected with the correspondence $Y'\leftrightarrow Z'$.

Theorem 3.3: (a) Fix a regular semisimple F_0 and $P, P_x \in [F_0,\mathfrak{g}]$. Set $F = \epsilon^{-1}F_0$. Then $u = P + O(\epsilon)$ as $\epsilon \to 0$. (b) Fix a regular semisimple $F \in \mathfrak{g}$ and $P_0, P_{0,x} \in [F,\mathfrak{g}]$. Set $P = \nu P_0$, $P_x = \nu P_{0x}$. Then $[F,u] = \nu([F,P_0] - P_{0,x}) + O(\nu^2)$, as $\nu \to 0$ (equivalently, $u = \nu(P_0 - (\mathrm{ad}\, F)^{-1} P_{0,x}) + O(\nu^2)$, where $(\mathrm{ad}\, F)^{-1}$ is an isomorphism on $[F,\mathfrak{g}] \ni P_{0,x}$). (c) Fix a regular semisimple $F \in \mathfrak{g}$ and let $\mathcal{J} = \mathfrak{g}^F$ be the corresponding Cartan subalgebra with root system Δ. Then the determinant of the Fréchet derivative of the linearization of the map $(P,P_x) \to u$, is given by

$$\prod_{\alpha \in \Delta} (1 - \alpha(F)^{-1}\partial). \tag{3.9}$$

Proof: (a) In all three cases we use the fact that by Theorem 3.2, u is rational in the parameters involved: in the present case, as a function of ϵ. Let us write then $u = \epsilon^s(u_0 + O(\epsilon))$ with some $s \leqslant 0$ and require $u_0 \neq 0$ for $s < 0$ (we thus allow $u_0 = 0$ for $s = 0$, taking care of the possibility of u having positive s-asymptotics in ϵ). Rewriting (3.2) in long hand, we have

$$[\mathrm{ad}(\epsilon^{-1}F_0 - P)]^2[P - \epsilon^s(u_0 + O(\epsilon))]$$
$$= [\mathrm{ad}(\epsilon^{-1}F_0 - P)]P_x = O(\epsilon^{-1}) = O(\epsilon^{s-2}).$$

This yields

$$(\mathrm{ad}\, F_0)^2 u_0 = 0 \quad \text{if} \quad s < 0,$$
$$(\mathrm{ad}\, F_0)(P - u_0) = 0 \quad \text{if} \quad s = 0.$$

But ad F_0 is nonsingular on $[F_0, \mathfrak{g}]$ which forces $u_0 = 0$ for $s < 0$, a contradiction with the choice $u_0 \neq 0$ made for $s < 0$. Thus $s = 0$ and $P - u_0 = 0$, proving (a). (b) For $\nu \to 0$, write $u = \nu^s(u_0 + O(\nu))$, where $s \leqslant 1$ and $u_0 \neq 0$ for $s < 1$. Then, as above, (3.2) can be rewritten as

$$[\mathrm{ad}(F - \nu P_0)^2](\nu P_0 - u) = \nu[F, P_{0,x}] + O(\nu^2),$$

which yields

$$(\mathrm{ad}\, F)^2 u_0 = 0, \quad \text{if} \quad s < 1,$$
$$(\mathrm{ad}\, F)^2(P_0 - u_0) = (\mathrm{ad}\, F)P_{0,x} \quad \text{if} \quad s = 1,$$

which forces, as above, $s = 1$ and $P_{0,x} = [F, P_0 - u_0]$. (c) If $u = f(P,P_x)$ is a locally smooth map with $0 = f(0,0)$, its linearization $l(f)$ (at zero) is defined by $\bar{u} = l(f)(P,P_x) := (d/d\nu)\, f(\nu P, \nu P_x)|_{\nu=0}$. By (b), we get $\bar{u} = P_0 - (\mathrm{ad}\, F)^{-1} P_{0,x}$. Choose a nonzero vector E_α in the root space $\mathfrak{g}_\alpha$ for every root $\alpha \in \Delta$. Then we can rewrite the above formula as

$$\bar{u}_\alpha = P_{0\alpha} - \alpha(F)^{-1}P_{0\alpha,x}. \tag{3.10}$$

Now recall that the Fréchet derivative of any map ϕ given by a nonlinear differential operator of the form $u_i = u_i(v_j^{(m)})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant N$, is an $n \times N$ matrix $D(\phi)$ of differential operators, defined by

$$D(\phi)_{ij} = \Sigma_{m>0} \frac{\partial u_i}{\partial v_j^{(m)}} \partial^m, \quad \partial = \frac{d}{dx}.$$

Thus for the case (3.10), we get the matrix-diag(..., $1 - \alpha(F)^{-1}\partial$,...), which implies (3.9).

Q.E.D.

IV. EVOLUTION EQUATIONS

In this section we construct deformations of Eqs. (2.4). The idea is to interpret them as defined on $\mathscr{H}''$ rather than on $\mathfrak{g}$.

We begin by recalling some simple notions from the calculus. Suppose N is a manifold (smooth, like everything else in this section), let $D(N)$ denote the set of all vector fields on N [that is, derivations of $C^\infty(N)$.] Consider separately $\mathbb{R}^1$ with the coordinate x and vector field d/dx. If $X \in D(N)$, a (local) trajectory of X is a map $\gamma: I^1 \to N$ such that

$$\frac{d}{dx}\gamma^* = \gamma^* X, \quad I^1 \text{ is an interval in } \mathbb{R}^1, \tag{4.1}$$

understood as the equality of operators on $C^\infty(N)$ [with values in $C^\infty(I^1)$]. If one chooses a (local) coordinate system $(y_1,\ldots,y_n)$ on N and if $X = \Sigma g_i(y)\partial/\partial y_i$, then (4.1) is equivalent to the familiar form of a system of ODE's:

$$\frac{d\gamma^*(y_i)}{dx} = \gamma^*(g_i), \quad 1 \leqslant i \leqslant n. \tag{4.2}$$

If the field X depends upon parameters $\mu_1,\ldots,\mu_k$ (x may be one of them), definition (4.2) works equally well.

What we need is a bit more. Suppose X and Y are two families of vector fields on N, and they both depend upon two parameters which we denote x and t. Consider $I^2 = I^1 \times I^1$ with these coordinates x and t and fix two vector fields $\partial = \partial/\partial x$ and $\partial_t = \partial/\partial t$ on I^2.

Definition 4.1: A trajectory of the pair X, Y is a map $\gamma: I^2 \to N$ such that

$$\frac{\partial}{\partial x}\gamma^* = \gamma^* X, \tag{4.3}$$

$$\frac{\partial}{\partial t}\gamma^* = \gamma^* Y. \tag{4.4}$$

Definition–Proposition: Let $X = X(\mu)$ be a family of vector fields on N depending upon parameters $\mu = (\mu_1,\ldots,\mu_k)$. Then the set of operators $X_{\mu_i} := \dfrac{\partial X}{\partial \mu_i}$, $1 \leqslant i \leqslant k$, defined by

$$X_{\mu_i}(h) = \frac{\partial}{\partial \mu_i} X(h), \quad h \in C^\infty(N),$$

is again a family of vector fields on N with parameters μ.

Proof: Differentiate, with respect to μ_i, the equality

$$X(h_1h_2) = h_1X(h_2) + h_2X(h_1), \quad h_1,h_2 \in C^\infty(N).$$

Proposition 4.2: A trajectory γ for the pair X, Y can be drawn through every point $(x,t,n) \in I^2 \times N$ [i.e., $\gamma(x,t) = n$] if and only if

$$[X,Y] - X_t + Y_x = 0, \tag{4.5}$$

as operator on $C^\infty(N)$.

Proof: The only integrability condition for γ is

$$\frac{\partial}{\partial t}(4.3) = \frac{\partial}{\partial x}(4.4),$$

as follows, e.g., from writing (4.3), (4.4) in local coordinates. We have then

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\gamma^* = \frac{\partial}{\partial t}\gamma^* X = (\gamma^* Y)X + \gamma^* X_t,$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\gamma^* = \frac{\partial}{\partial x}\gamma^* Y = (\gamma^* X)Y + \gamma^* Y_x,$$

so

$$\gamma^*([X,Y] - X_t + Y_x) = 0.$$

Q.E.D

We remark in passing how to transform (4.5) into a more familiar form. Consider $\bar{N} := N \times I^2$ and extend naturally on $\bar{N}$ vector fields X, Y, ∂ and ∂_t, continuing to denote them by the same symbols. Then (4.5) can be written as

$$[\partial + X, \partial_t + Y] = 0. \tag{4.6}$$

It is obvious now how to proceed. Rewrite (2.4) as

$$[-U, -V] - (-U_t) + (-V_x) = 0, \tag{4.7}$$

and consider the representation of $\mathfrak{g}$ in $D(\mathscr{H}'')$, as in Sec. III. Applying proposition 4.2, we see that (4.7) is the integrability condition for the system

$$\frac{\partial}{\partial x}\gamma^* = \gamma^*(-X_{\lambda F - u}), \tag{4.8}$$

$$\frac{\partial}{\partial t}\gamma^* = \gamma^*(-X_V), \tag{4.9}$$

for $u = u(x,t)$ fixed. Applying Theorem 3.2 to Eq. (4.8), we find u (or rather $-u$) as a rational function of $P(\gamma)$. This enables us to eliminate u in favor of γ in (4.9) which thus becomes our deformed equation, if we define the deformation parameter ϵ as $-\lambda^{-1}$. Indeed, Theorem 3.3(a) then yields that $-u = P(\gamma) + O(\epsilon)$, thus $-P(\gamma)$ (or $-\gamma$) is the deformed variable [analog of w in (1.3)].

Proposition 4.3: Deformed Eq. (4.9) is not equivalant to undeformed one (2.4) under any change of variables.

Proof: Use existence of the reduction (4.8). If (4.9) were equivalent to (2.4), the map $u \mapsto \gamma$ which inverts (4.8) (understood as a rational map $\gamma \mapsto u$ by Theorem 3.2) will be a (finite) differential operator. In particular, its linearization will invert the linearization of (4.8). Therefore the Fréchet derivative of this linearization will invert the Fréchet derivative of linearized (4.8). Taking determinants, we get a differential operator which inverts expression (3.9), which is a differential operator of positive order [since $\alpha(F) \not\equiv 0$ for $\alpha \in \Delta$], a contradiction.

Q.E.D.

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[1] R. M. Miura, C. S. Gardner, and M. D. Kruskal, "Korteweg–de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion," J. Math. Phys. **9**, 1204–1209 (1968).
[2] Yu. I. Manin, "Algebraic Aspects of Nonlinear Differential Equations," J. Sov. Math. **11**, 1–122 (1979).
[3] V. G. Drinfel'd and V. V. Sokolov, "Korteweg–de Vries Type Equations and Simple Lie Algebras," Dokl. Acad. Nauk SSSR **258**, 11–16 (1981) (in Russian).
[4] G. Wilson, "Commuting Flows and Conservation Laws for Lax Equations," Math. Proc. Cambridge Philos. Soc. **86**, 131–143 (1979).
[5] G. Wilson, "The Modified Lax and Two-Dimensional Toda Lattice Equations Associated with Simple Lie Algebras," Ergodic Theory Dynamical Systems **1**, 361–380 (1982).
[6] B. A. Kupershmidt, "On the Nature of the Gardner Transformation," J. Math. Phys. **22**, 449–451 (1981).
[7] B. A. Kupershmidt, "Deformations of Integrable Systems," Proc. R. Ir. Acad. Sect. A (to appear).

Higher-order parabolic approximations to time-independent wave equations

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A sequence of numerically tractable higher-order parabolic approximations is derived for the reduced wave equation in an inhomogeneous medium. The derivation is motivated by a definition of waves propagating in a distinguished direction. For a homogeneous medium these definitions are exact and yield uncoupled, infinite-order parabolic equations which are equivalent to the wave equation. The difficulty of obtaining higher-order parabolic approximations for the elastic wave equation in an inhomogeneous medium is also discussed.

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I. INTRODUCTION

This paper presents the derivation of a sequence of increasingly accurate parabolic approximations to the reduced wave equation. The derivation has two parts. First, a distinguished direction is assumed to be defined by the excitation of the medium (e.g., by an incident pencil of light) and functions that approximate waves propagating in both senses of this direction are defined. The sum of these functions is the full field within the medium. Next, a pair of coupled equations for these functions is derived. This pair is a first-order system in the distinguished direction and is equivalent to the reduced wave equation. Parabolic approximations result from neglecting the coupling terms in this pair.

The problem of identifying waves propagating in a given direction in a nonuniform medium was first discussed by Bremmer.[1] Motivated by this work a rederivation of the Leontovich–Fock approximation,[2] and several related approximations, were given in Ref. 3. The most accurate parabolic approximation to the Helmholtz equation was derived in Ref. 4. The results derived here, though less accurate, appear to be more amenable to numerical integration than those in Ref. 4.

Parabolic approximations to the elastic wave equation are also discussed. Further justification for some of the results in Ref. 5 is given, and the difficulty of obtaining higher parabolic approximations to the elastic wave equation is discussed.

II. DEFINITION OF WAVES PROPAGATING IN A DISTINGUISHED DIRECTION

Consider the Helmholtz equation

$$\Delta\Psi(\mathbf{x}) + k^2(\mathbf{x})\Psi(\mathbf{x}) = 0 \tag{2.1}$$

when $k^2(\mathbf{x}) = k_0^2$, k_0^2 const. Solutions of (2.1) can be written

$$\Psi(\mathbf{x}) = \int e^{i\mathbf{p}\cdot\mathbf{x}} A(\mathbf{p})\delta(p^2 - k_0^2)\, d^3\mathbf{p}. \tag{2.2}$$

Equation (2.2) expresses $\Psi(\mathbf{x})$ as a superposition of plane waves. Each plane wave component satisfies (2.1) with $p^2 = k_0^2$. The function $A(\mathbf{p})$ depends upon the boundary conditions of the problem. If the boundary conditions single out a particular direction of propagation (e.g., an incident pencil of light singles out the direction along the incident beam), it is of interest to define functions that are good approximations to the components of the wave propagating in either sense of the distinguished direction.

Assume the distinguished direction is the x_1 axis and let p_1 denote the x_1 component of momentum and $\mathbf{R} = (p_2, p_3)$ denote the momentum transverse to p_1. For the boundary conditions of interest the amplitude factor $A(\mathbf{p})$ will contain only small contributors due to $\mathbf{R}$. Thus an approximate decomposition of $\Psi(\mathbf{x})$ into upward and downward waves is sought when $A(\mathbf{p})$ is a sharply peaked function of p_1.

To obtain this decomposition first define $\mathbf{x}_\perp = (x_2, x_3)$ and in (2.2) integrate with respect to p_1. The result is

$$\Psi(\mathbf{x}) = \int \{A^+ \exp[i(k_0^2 - R^2)^{1/2}x_1] + A^- \exp[-i(k_0^2 - R^2)^{1/2}x_1]\} e^{i\mathbf{R}\cdot\mathbf{x}_\perp} d^2\mathbf{R}. \tag{2.3}$$

This representation of Ψ suggests the definition

$$\Psi^\pm(\mathbf{x}) = \int A^\pm \exp[\pm i(k_0^2 - R^2)^{1/2}x_1] e^{i\mathbf{R}\cdot\mathbf{x}_\perp} d^2\mathbf{R}, \tag{2.4}$$

where $\Psi^\pm$ are the upward and downward components, respectively, of Ψ.

Now from Eq. (2.3) it follows that

$$\Psi_{x_1}(\mathbf{x}) = \int i(k_0^2 - R^2)^{1/2}\{A^+ \exp[i(k_0^2 - R^2)^{1/2}x_1] - A^- \exp[-i(k_0^2 - R^2)^{1/2}x_1]\} e^{i\mathbf{R}\cdot\mathbf{x}_\perp} d^2\mathbf{R}, \tag{2.5}$$

where the subscript x_1 denotes partial differentiation. By assumption, the region of interest is $p_1^2 \simeq k_0^2$; i.e., $R^2/k_0^2 \ll 1$. Thus,

$$(k_0^2 - R^2)^{1/2} = k_0(1 - R^2/2k_0^2 - R^4/8k_0^4 - \cdots). \tag{2.6}$$

Using the fact that

$$\int R^{2N} A^\pm \exp[\pm i(k_0^2 - R^2)^{1/2}x_1] e^{i\mathbf{R}\cdot\mathbf{x}_\perp} d^2\mathbf{R} = -(\Delta_\perp)^N \int A^\pm \exp[\pm i(k_0^2 - R^2)^{1/2}x_1] e^{i\mathbf{R}\cdot\mathbf{x}_\perp} d^2\mathbf{R}, \tag{2.7}$$

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where

$$\Delta_\perp = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{2.8}$$

it follows that Eq. (2.5) can be written as

$$\frac{1}{ik_0}\Psi_{x_1}(\mathbf{x}) = \left(1 + \frac{1}{k_0^2}\Delta_\perp\right)^{1/2} \int \{A^+ \exp[i(k_0^2 - R)^{1/2}x_1] - A^- \exp[-i(k_0^2 - R^2)^{1/2}x_1]\} e^{i\mathbf{R}\cdot\mathbf{x}_\perp} d^2\mathbf{R}. \tag{2.9}$$

Combining Eqs. (2.3) and (2.9) and using Eq. (2.4), it is now seen that

$$\Psi^\pm(\mathbf{x}) = \tfrac{1}{2}\{\Psi \pm (1/ik_0)[1 + (1/k_0^2)\Delta_\perp]^{-1/2}\Psi_{x_1}\}. \tag{2.10}$$

Equation (2.10) provides a technique for approximating $\Psi^\pm$. Let S_N^{-1} denote the first $N+1$ terms in the formal series expansion of $[1 + (1/k_0^2)\Delta_\perp]^{-1/2}$ and define

$$\Psi_N^\pm = \tfrac{1}{2}[\Psi \pm (1/ik_0)S_N^{-1}\Psi_z] = D_N^\pm \Psi. \tag{2.11}$$

Now the operators $D_N^\pm$ approximately project out the $\pm$ components of Ψ so that $\Psi_N^\pm \simeq \Psi^\pm$ with the accuracy of the approximation increasing as $N \to \infty$.

If $k^2(\mathbf{x}) = k_0^2(1 + \eta(\mathbf{x}))$ and $\eta(\mathbf{x})$ is small and slowly varying, the definitions (2.11) should continue to represent a good approximate decomposition of Ψ into upward and downward components. A more nearly exact sequence of definitions of $\pm$ components would be obtained from the expansion of $[k^2(x) + \Delta_\perp]^{-1/2}$.[4] However, due to the lack of commutivity of $k^2(\mathbf{x})$ and $\Delta_\perp$, it is not apparent how to explicitly express these definitions short of a spectral analysis of $[k^2(x) + \Delta_\perp]^{-1/2}$.

Having settled on definitions of $\Psi^\pm$ for a medium with small and slowly varying inhomogeneities, the next step is to obtain coupled equations for the components. This is done in the following section.

III. DERIVATION OF HIGHER ORDER PARABOLIC APPROXIMATIONS

From the definitions of $\Psi^\pm$ given in Eq. (2.11), it follows that

$$\Psi = \Psi_N^+ + \Psi_N^- \tag{3.1}$$

and

$$\Psi_{x_1} = ik_0 S_N(\Psi_N^+ - \Psi_N^-) \tag{3.2}$$

for $N = 0,1,2,\cdots$. In Eq. (3.2), S_N denotes the first $N+1$ terms in the formal series expansion of $[1 + (1/k_0^2)\Delta_\perp]^{1/2}$. To simplify the notation, the subscript N will be deleted from ensuing calculations in this section. The final results can be interpreted by appending subscript N to all functions $\Psi^\pm$ and operators S.

Differentiating Eq. (3.2) with respect to x_1 and using Eq. (2.1) to eliminate $\partial^2\Psi/\partial x_1^2$ yields

$$ik_0 \frac{\partial}{\partial x_1} S(\Psi^+ - \Psi^-) = -[k^2(\mathbf{x}) + \Delta_\perp](\Psi^+ + \Psi^-). \tag{3.3}$$

Operating on Eq. (3.2) with $ik_0 S$ produces

$$ik_0 \frac{\partial}{\partial x_1} S(\Psi^+ - \Psi^-) = -k_0^2 S^2(\Psi^+ - \Psi^-). \tag{3.4}$$

Adding and subtracting Eqs. (3.3) and (3.4) and rearranging terms yields the system

$$\left[2ik_0 \frac{\partial}{\partial x_1} S + k^2(x) + \Delta_\perp + k_0^2 S^2\right]\Psi^+ = -[k^2(\mathbf{x}) + \Delta_\perp - k_0^2 S^2]\Psi^-, \tag{3.5a}$$

$$\left[2ik_0 \frac{\partial}{\partial x_1} S - k^2(\mathbf{x}) - \Delta_\perp - k_0^2 S^2\right]\Psi^- = [k^2(\mathbf{x}) + \Delta_\perp - k_0^2 S^2]\Psi^+. \tag{3.5b}$$

The system (3.5) is equivalent to Eq. (2.1), and has the advantage that a parabolic approximation to the system is easily justified. To see this, display the approximate phases of $\Psi^\pm$ by writing

$$\Psi^\pm = v^\pm e^{\pm ik_0 x_1}. \tag{3.6}$$

Substituting (3.6) into Eq. (3.5) yields

$$\left[2ik_0 \frac{\partial}{\partial x_1} S + k^2(\mathbf{x}) + \Delta_\perp - 2k_0^2 S + k_0^2 S^2\right]v^+ = -[k^2(\mathbf{x}) + \Delta_\perp - k_0^2 S^2]e^{-2ik_0 x_1}v^-, \tag{3.7a}$$

$$\left[2ik_0 \frac{\partial}{\partial x_1} S - k^2(\mathbf{x}) - \Delta_\perp + 2k_0^2 S - k_0^2 S^2\right]v^- = [k^2(\mathbf{x}) + \Delta_\perp - k_0^2 S^2]e^{2ik_0 x_1}v^+. \tag{3.7b}$$

The parabolic approximations for $v^\pm$ are obtained by suppressing the reflection terms in Eq. (3.7); i.e., replacing the right-hand sides of the equations with 0. This approximation is valid since $v^\pm$ and $k^2(\mathbf{x})$ vary slowly relative to $e^{\pm ik_0 x_1}$. Thus, the parabolic approximations for $v^\pm$ are

$$\left[2ik_0 \frac{\partial}{\partial x_1} S \pm k_0^2 \eta(\mathbf{x}) \pm \Delta_\perp \pm k_0^2 (S-1)^2\right]v^\pm = 0. \tag{3.8}$$

If S is chosen to be S_0,

$$S = S_0 = 1,$$

then Eq. (3.8) becomes the Fock approximation

$$\left[2ik_0 \frac{\partial}{\partial x_1} \pm k_0^2 \eta(\mathbf{x}) \pm \Delta_\perp\right]v_0^\pm = 0. \tag{3.9}$$

To obtain a higher order approximation, set

$$S = S_1 = 1 + (1/2k_0^2)\Delta_\perp,$$

which yields

$$\left[2ik_0 \frac{\partial}{\partial x_1} \pm k_0^2 \eta(\mathbf{x}) \pm \Delta_\perp + \frac{i}{k_0}\Delta_\perp \frac{\partial}{\partial x_1} \pm \frac{1}{4k_0^2}(\Delta_\perp)^2\right]v_1^\pm = 0. \tag{3.10}$$

In the limit $N \to \infty$, with

$$S = S_\infty = (1 + \Delta_\perp/k_0^2)^{1/2},$$

the systems (3.5) and (3.7) become

$$\left[2ik_0 \frac{\partial}{\partial x_1} S_\infty \pm 2k_0^2 \pm k_0^2 \eta(\mathbf{x}) \pm 2\Delta_\perp\right]\Psi_\infty^\pm = \mp k_0^2 \eta(\mathbf{x})\Psi_\infty^\mp \tag{3.11}$$

and

$$\left[2ik_0\frac{\partial}{\partial x_1}S_\infty \pm 2k_0^2 \pm k_0^2\eta(\mathbf{x}) \mp 2k_0^2 S_\infty \pm 2\Delta_\perp\right]v_\infty^\pm = \mp k_0^2\eta(\mathbf{x})e^{\mp 2ik_0x_1}v_\infty^\mp. \tag{3.12}$$

For a homogeneous medium, $\eta\equiv 0$, and the equations uncouple, yielding

$$\left(ik_0\frac{\partial}{\partial x_1}S_\infty \pm k_0^2 \pm \Delta_\perp\right)\Psi_\infty^\pm = 0$$

and

$$\left(ik_0\frac{\partial}{\partial x_1}S_\infty \pm k_0^2 \mp k_0^2 S_\infty \pm \Delta_\perp\right)v_\infty^\pm = 0.$$

IV. THE ELASTIC WAVE EQUATION

The time independent elastodynamic wave equation for an isotropic medium is

$$(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) + (\nabla\lambda)(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) + (\nabla\mu)\times(\nabla\times\mathbf{u}) + 2[(\nabla\mu)\cdot\nabla]\mathbf{u} + \rho\omega^2\mathbf{u} = \mathbf{0}, \tag{4.1}$$

where λ and μ are the Lamé parameters, ρ is the mass density, $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the elastic wave displacement, and $\mathbf{x} = (x_1,x_2,x_3)\in R^3$ is a point in the medium. If λ,μ, and ρ are constants then solutions of (4.1) are given by

$$\mathbf{u}(\mathbf{x}) = \iiint\left(\frac{1}{k_L^2}[\mathbf{A}(\mathbf{p})\cdot\mathbf{p}]\mathbf{p}\delta(p^2-k_L^2) + \left\{\mathbf{A}(\mathbf{p}) - \frac{1}{k_T^2}[\mathbf{A}(\mathbf{p})\cdot\mathbf{p}]\mathbf{p}\right\}\delta(p^2-k_T^2)\right)e^{i\mathbf{p}\cdot\mathbf{x}}\,d^3\mathbf{p}, \tag{4.2}$$

where

$$k_L^2 = \rho\omega^2/(\lambda+2\mu),$$
$$k_T^2 = \rho\omega^2/\mu,$$

and

$$p^2 = \mathbf{p}\cdot\mathbf{p}.$$

As in Sec. II, $\mathbf{A}(\mathbf{p})$ depends on the boundary conditions of the problem. The integral of the first term in the integrand in (4.2) is the longitudinal wave $\mathbf{u}_L$ and the second term yields the transverse wave $\mathbf{u}_T$. Assuming the distinguished direction of propagation to the parallel to the x_1 axis, Eq. (4.2) yields

$$\mathbf{u}_L = \mathbf{u}_L^+ + \mathbf{u}_L^-, \tag{4.3}$$

where

$$\mathbf{u}_L^\pm(\mathbf{x}) = \iint\mathbf{A}_L^\pm((k_L^2-R^2)^{1/2},p_2,p_3) \times\exp\{\pm i[(k_L^2-R^2)^{1/2}x_1 + p_2x_2 + p_3x_3]\}\,dp_2dp_3, \tag{4.4}$$

and

$$\mathbf{A}_L^\pm(p_1,p_2,p_3) = \tfrac{1}{2}(1/k_L^2\sqrt{k_L^2-R^2}) [\mathbf{A}(\pm p_1,p_2,p_3)\cdot(\pm p_1,p_2,p_3)](\pm p_1,p_2,p_3), \tag{4.5}$$

and

$$R^2 = p_2^2 + p_3^2.$$

In this representation, $\mathbf{u}_L^+$ is the upward or forward moving component of the longitudinal wave, and $\mathbf{u}_L^-$ is the downward moving component. Similarly,

$$\mathbf{u}_T = \mathbf{u}_T^+ + \mathbf{u}_T^-, \tag{4.6}$$

where

$$\mathbf{u}_T^\pm(\mathbf{x}) = \iint\mathbf{A}_T^\pm([k_T^2-R^2]^{1/2},p_2,p_3) \times\exp\{\pm i[(k_T^2-R^2)^{1/2}x_1 + p_2x_3 + p_3x_3]\}\,dp_2dp_3, \tag{4.7}$$

and

$$\mathbf{A}_T^\pm(p_1,p_2,p_3) = \tfrac{1}{2}(1/\sqrt{k_T^2-R^2}) \times[\mathbf{A}(\pm p_1,p_2,p_3) - (1/k_T^2)(\mathbf{A}(\pm p_1,p_2,p_3) \cdot(\pm p_1,p_2,p_3))(\pm p_1,p_2,p_3)]. \tag{4.8}$$

In the same manner as in Sec. II, the $\pm$ components of $\mathbf{u}_L$ and $\mathbf{u}_T$ can be written as

$$\mathbf{u}_L^\pm = \tfrac{1}{2}\left(\mathbf{u}_L \pm \frac{1}{ik_L}S_L^{-1}\frac{\partial\mathbf{u}_L}{\partial x_1}\right), \tag{4.9}$$

$$\mathbf{u}_T^\pm = \tfrac{1}{2}\left(\mathbf{u}_T \pm \frac{1}{ik_T}S_T^{-1}\frac{\partial\mathbf{u}_T}{\partial x_1}\right). \tag{4.10}$$

The operator S_L is given by

$$S_L = [1+(1/k_L^2)\Delta_\perp]^{1/2} = \lim_{N\to\infty}S_{L,N}, \tag{4.11}$$

where $S_{L,N}$ is the first $N+1$ terms in the formal series expansion of S_L. Also,

$$S_L^{-1} = [1+(1/k_L^2)\Delta_\perp]^{-1/2} = \lim_{N\to\infty}S_{L,N}^{-1}. \tag{4.12}$$

The operator $S_{T,N}$ is defined similarly, with k_T replacing k_L in Eqs. (4.11) and (4.12). Notice that Eqs. (4.9) and (4.10) imply that

$$ik_LS_L(\mathbf{u}_L^+ - \mathbf{u}_L^-) = \frac{\partial}{\partial x_1}(\mathbf{u}_L^+ + \mathbf{u}_L^-), \tag{4.13}$$

$$ik_TS_T(\mathbf{u}_T^+ - \mathbf{u}_T^-) = \frac{\partial}{\partial x_1}(\mathbf{u}_T^+ + \mathbf{u}_T^-). \tag{4.14}$$

In the same manner as in Sec. III, Eqs. (4.9) and (4.10) are used as definitions of $\mathbf{u}_L^\pm$ and $\mathbf{u}_T^\pm$ for the case when λ,μ, and ρ are not constant. Write

$$\lambda(\mathbf{x}) = \lambda_0(1+\tilde{\lambda}(\mathbf{x})), \tag{4.15}$$

$$\mu(\mathbf{x}) = \mu_0(1+\tilde{\mu}(\mathbf{x})), \tag{4.16}$$

$$\rho(\mathbf{x}) = \rho_0(1+\tilde{\rho}(\mathbf{x})), \tag{4.17}$$

where λ_0, μ_0, and ρ_0 are constants, and define

$$k_L^2 = \rho_0\omega^2/(\lambda_0+2\mu_0),$$
$$k_T^2 = \rho_0\omega^2/\mu_0$$

as the constants which appear in Eqs. (4.9)–(4.12). The functions $\tilde{\lambda},\tilde{\mu},\tilde{\rho}$ in Eqs. (4.15)–(4.17) are assumed to be small and slowly varying.

Now write

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \tag{4.18}$$

where **i**,**j**,**k** are unit vectors along the x_1,x_2,x_3 axes. Since the distinguished direction of propagation is parallel to **i**, assume that

$$\mathbf{u}_L \approx u_1\mathbf{i} \tag{4.19}$$

and

$$\mathbf{u}_T \approx u_2\mathbf{j} + u_3\mathbf{k}. \tag{4.20}$$

Similarly, assume that

$$\mathbf{u}_L^{\pm} \approx u_1^{\pm}\mathbf{i}, \tag{4.21}$$

$$\mathbf{u}_T^{\pm} \approx u_2^{\pm}\mathbf{j} + u_3^{\pm}\mathbf{k}, \tag{4.22}$$

where

$$u_n = u_n^+ + u_n^-$$

for $n = 1,2,3$. Now if the approximations

$$S_L \approx S_{L,0} = 1,$$

$$S_T \approx S_{T,0} = 1$$

and (4.21) and (4.22) are used, then Eqs. (4.13) and (4.14) become

$$ik_L(u_1^+ - u_1^-) = \frac{\partial}{\partial x_1}(u_1^+ + u_1^-),$$

$$ik_T(u_j^+ - u_j^-) = \frac{\partial}{\partial x_1}(u_j^+ + u_j^-), \quad j = 2,3,$$

which is the (T_0) splitting introduced in Ref. 5, Sec. 2. This provides additional justification for that splitting, along with the heuristics given in Ref. 5, Sec. 3. A careful examination of (4.5) and (4.8) shows that the assumptions (4.19) and (4.20) are valid only to $O(k_L^{-1})$ and $O(k_T^{-1})$. Consequently, the approximations to $\mathbf{u}_L^{\pm}$ and $\mathbf{u}_T^{\pm}$ will not be improved by considering higher-order approximations to S_L and S_T in (4.13) and (4.14) such as $S_{L,1}$, $S_{T,1}$, etc.

V. CONCLUSION

The system of equations (3.5) [or (3.7)] has two types of coupling terms. There are couplings due to $\eta(\mathbf{x})$ (the inhomogeneity of the medium) and those due to k_0^2 and $\Delta_\perp$ and their combinations. These later terms are present even if $\eta(\mathbf{x})$ is zero. Thus, for finite N, couplings (interpreted as reflections) exist even in homogeneous media. This unphysical feature is an artifact of the approximate identification of upward and downward waves. In the $N\to\infty$ limit, Eq. (3.12) results. In this case the only coupling is due to inhomogeneities in the medium. This is an important feature of (3.12) shared with the results in Ref. 4. It appears that (3.11) and (3.12), though less accurate than the parabolic approximation in Ref. 4, are more tractable. Both results fully take into account what might be called the kinematics of the propagation, i.e., those factors which for finite N yield the spurious (or "kinematical") reflections discussed above.

Notice that (3.12) can formally be written

$$\left[2ik_0\frac{\partial}{\partial x_1} \mp 2k_0^2 \pm 2k_0^2 S_\infty \pm k_0^2 S_\infty^{-1}\eta(\mathbf{x})\right]v_\infty^{\pm}$$
$$= \mp k_0^2 S_\infty^{-1}[\eta(\mathbf{x})\exp(\pm 2ik_0x_1)v_\infty^{\mp}]. \tag{5.1}$$

If $\eta(\mathbf{x}) = 0$, this becomes

$$\left[i\frac{\partial}{\partial x_1} \pm (k_0^2 + \Delta_\perp)^{1/2}\right]\Psi^{\pm}(x) = 0. \tag{5.2}$$

This agrees with a result in Ref. 6 for the homogeneous case and with Ref. 4.

The results for the elastic case are much less satisfactory. The reason for this is that it is necessary to decompose **u** into polarization components as well as upward and downward components for each polarization. This prevents the derivation of higher parabolic approximations. However, the results presented here further justify the derivations in Ref. 5 of equations that describe the dominant features of propagation of elastic waves in media with small and slowly varying Lamé parameters.

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[1] H. Bremmer, Comm. Pure Appl. Math. **4**, 105 (1951).
[2] V. A. Fock, *Electromagnetic Diffraction and Propagation Problems* (McMillan, New York, 1960).
[3] J. Corones, J. Math. Anal. Appl. **27**, 282 (1969).
[4] F. D. Tappert, in *Wave Propagation and Underwater Acoustics*, edited by J. B. Keller and J. S. Papadakis (Springer-Verlag, New York, 1977).
[5] J. Corones, B. DeFacio, and R. Krueger, J. Math. Phys. **23**, 577 (1982).
[6] J. Claerbout, *Fundamentals of Geophysical Data Processing* (McGraw-Hill, New York, 1976).

Simple waves in quasilinear hyperbolic systems [a]. I. Theory of simple waves and simple states. Examples of applications

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This paper presents a new method of construction of solutions to nonlinear, nonelliptic systems of partial differential equations and especially nonhomogeneous ones. These equations have been considered from the point of view of integral elements. In particular the connections between the structure of the set of integral elements and the possibility of a construction of special classes of solutions have been studied. These classes consist of what is called simple waves and k waves (for homogeneous systems) and simple states (in the case of nonhomogeneous systems). They provide us with a possibility for a selection of simple integral elements from the set of all integral elements. Analyses have been performed using differential forms and Cartan theory of system in involution. The problem has been reduced to examining Pfaff forms. The Cauchy problem for Pfaff systems has been formulated and solved using the Riemann function. Some remarks concerning the notion of Bäcklund transformations for the case of k waves have been formulated. It is shown that, in contrast to simple wave, the simple state has no gradient catastrophy. The technique presented of constructing the solutions in form of simple states has been illustrated by the examples of Korteweg and de Vries and four-dimensional Klein–Gordon, sine–Gordon, and Liouville equations. It has been shown that the known soliton equations are closely connected with the elliptical functions and especially with the P–Weierstrass functions.

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I. INTRODUCTION

This paper is a generalization and continuation of previous papers[1,2] concerning nonelliptical system of partial differential equations (p.d.e.) of the first order with the analytical coefficients of the type

$$a_j^{s\mu}(u^1,\ldots,u^l)\frac{\partial}{\partial x^\mu}u^j(x)=b^s(u^1,\ldots,u^l)\,, \tag{1.1}$$

where

$x=(x^1,\ldots,x^n)\in\mathscr{E}$, $u(x)=(u^1(x),\ldots,u^l(x))\in H$,

$s=1,\ldots,m$ is the number of equations,

$\mu=1,\ldots,n$ is the number of independent variables,

$j=1,\ldots,l$ is the number of unknown functions.

The system (1.1) is a nonhomogeneous one with coefficients dependent on the unknown functions (even when $m\geqslant l$). The Euclidean space $\mathscr{E}$—the space of independent variables—is called the physical space, $\mathscr{E}=R^n$, while the space H—the space of values of dependent variables—is called the hodograph[3] space, $H=R^l$. Let us assume that the initial conditions for the system (1.1) are smooth, i.e., $u_0(x)\in C^1(R^{n-1})$.

We are looking for solutions describing the propagation and nonlinear interactions of waves which can be realized in the above systems.

Such solutions are of particular interest in such domains of physics as field theory, electrodynamics, mechanics of continuous media, plasma theory, quantum theory, and the theory of relativity. As is illustrated by the examples in the text, they cover a wide range of the wave phenomena arising in the presence of external forces. These phenomena are described by systems of equations of the form (1.1) or systems which can be reduced to that form by introducing new unknown functions.

The methodological approach accepted in this work is based on the generalization of Riemann invariants.[4–8] Lately this method has been extensively developed by M. Burnat[9–11] and next by Z. Peradzyński[12–16] and W. Zajączkowski.[17,18] The results obtained for the homogeneous systems were so promising that it seemed to be worthwhile to try to extend this method and check its effectiveness for the case of nonhomogeneous partial differential equations. This is in short the aim of the present paper.

II. HOMOGENEOUS AND NONHOMOGENEOUS SIMPLE INTEGRAL ELEMENTS

The starting point for this paper is to make an algebraization (according to papers[1,19,20]) of the considered system of Eqs. (1.1). Partial differential equations (1.1) can be written as follows:

$$\frac{\partial u^j}{\partial x^\mu}\in\{L^j_\mu : a_j^{s\mu}L^j_\mu=b^s\}\,.$$

Definition 1: The matrix L^j_μ satisfying the above conditions at a given point $u_0\in H$ we shall call an integral element of the system (1.1).

The matrix $L=\|\partial u^j/\partial x^\mu\|$ is a matrix of the tangent mapping[21] $du(x)$: $\mathscr{E}\to T_uH$ given by the formula:

$$\mathscr{E}\ni(\delta x^\mu)\mapsto(\delta u^j)\in T_uH,\quad\text{where}\quad \delta u^j=\frac{\partial u^j}{\partial x^\mu}\delta x^\mu\,.$$

Tangent mapping $du(x)$ determines an element of linear

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space $L(\mathscr{E}, T_u H)$, which may be identified with the tensor product $T_u H \otimes \mathscr{E}^*$ (where $\mathscr{E}^*$ is the space dual to $\mathscr{E}$, i.e., space of linear forms $\lambda: \mathscr{E} \to R^1$). As is well known[22] each element of this tensor product is a finite sum of simple tensors, i.e., $L = \gamma \otimes \lambda$, where $\lambda \in \mathscr{E}^*$ is a covariant vector and $\gamma \in T_u H$ is a vector tangent to H at the point u.

Definition 2: The integral element L^j_μ is called a simple element, if the rank $\|L^j_\mu\| = 1$, i.e., when the corresponding tensor is a simple one.

To determine a simple integral element L, we must find such a $\gamma \in T_u H$ and $\lambda \in \mathscr{E}^*$ which satisfy

$$a^{s\mu}_j \gamma^j \lambda_\mu = \begin{cases} 0 & \text{for a homogeneous system,} \\ b^s & \text{for a nonhomogeneous system,} \end{cases} \quad (2.1)$$
$$s = 1,...,m,\ \mu = 1,...,n,\ j = 1,...,l.$$

Simple elements of the system (2.1) will be denoted, respectively, $\gamma \otimes \lambda$ for the homogeneous system [that is, when $b(u) \equiv 0$], and $\gamma_0 \otimes \lambda^0$ for nonhomogeneous system [when $b(u) \not\equiv 0$]. The existence conditions for simple integral elements can be derived directly from their algebraic representation. Thus simple homogeneous elements are directly connected with the existence of characteristic vectors. Namely, the necessary and sufficient condition for the existence of a nonzero solution γ of Eqs. (2.1) is

$$\operatorname{rank}\|a^{s\mu}_j \lambda_\mu\| < l. \quad (2.2)$$

The relations (2.1) and (2.2) are called the wave relation and the dispersion relation,[13] respectively. If the covector λ satisfies the dispersion relation (2.2), then there exists a polarization vector $\gamma \in T_u H$, satisfying the wave relation (2.1). Thus there exists a relation between directions γ and λ. Conversely, the vector γ for which there exists a λ such that (2.1) holds will be called a characteristic vector.

The conditions of existence of the nonhomogeneous simple integral elements are determined by the Kronecker–Cappela theorem. Namely, the necessary conditions of existence of a nonzero solution γ_0 of the system (2.1) is

$$\operatorname{rank}\|(a^{s\mu}_j \lambda^0_\mu, b^s)\| = \operatorname{rank}\|(a^{s\mu}_j \lambda^0_\mu)\|. \quad (2.3)$$

For example, if $m = l$ and if the determinant $\det(a^{s\mu}_j \lambda^0_\mu) \neq 0$, we can determine the polarization vector γ_0 by

$$\gamma_0 = (a^{s\mu}_j \lambda^0_\mu)^{-1} b^s \quad (2.4)$$

and so the vector $\gamma_0 \in T_u H$ is the function of the variables $(u, \lambda^0) \in H \times \mathscr{E}^*$. Hence $H \times \mathscr{E}^* \ni (u, \lambda^0) \to \gamma_0(u, \lambda^0) \in T_u H$.

Now we will show that the simple integral elements can be used for constructing solutions of the system (1.1). These solutions can be interpreted as a propagation and nonlinear interaction of many simple waves on the simple state. For these purposes let us remind the notions of simple wave and simple state.

III. SIMPLE RIEMANN WAVES

Now let us introduce the notion of the simple wave and simple state. These notions will provide us with a tool for an extraction of the simple integral elements from the set of all integral elements. Let the mapping $u: D \to H$, $(D \subset \mathscr{E})$ be a solution of the system (1.1).

Definition 3: The solution of the system (1.1) is called a simple wave (for a homogeneous system) or a simple state (in the case of a nonhomogeneous system) if the tangent mapping $du(x)$ is a simple element at each point $x_0 \in D$.

In other words, the derived mapping [the tangent mapping $du(x)$] of a simple wave (or a simple state) is a simple integral element.

In that case it follows from the rank of the mapping theorem[22] that $u(D) \subset H$ is a one-dimensional submanifold, i.e., some curve Γ. Let $R \mapsto f(R) \in H$ be a parametrization of Γ. Then $u(x) = f(R(x))$, where $R(x)$ is a scalar function called Riemann-invariant. Since $\partial u^j / \partial x^\mu = (df^j/dR)\, \partial R / \partial x^\mu$, then for the homogeneous system (1.1) the tangent vector

$$\frac{d}{dR} f(R) = \gamma(R) \in T_{f(R)} H \quad (3.1)$$

is a characteristic vector. Let $\lambda(R)$ be a characteristic covector related to $\gamma(R)$. Since $u(x)$ satisfies Eqs. (1.1), covectors proportional to $dR(x)$ are related to $\gamma(R)$. Let us assume that the following equality holds:

$$dR(x) = \xi(x)\, \lambda(R(x)).$$

When $\operatorname{rank}\| a^{s\mu}_j \gamma^j \| = n - 1$, the above assumption is automatically satisfied. That is a typical (i.e., generic) case for the system when we have more equations than independent variables ($m > n$).

Now we prove that there exists a function φ of one variable such that the following equation holds: $R(x) = \varphi(\mathscr{O}(x))$, where $\mathscr{O}(x) = \lambda_\mu(R(x)) \cdot x^\mu$ is the phase of a simple wave. Indeed $d\mathscr{O}(x) = (1 + \xi d\lambda_\mu / dR \cdot x^\mu)\, \lambda_\mu\, dx^\mu$; thus $d\mathscr{O} \wedge dR = 0$, so that $R(x) = \varphi(\mathscr{O}(x))$ in the neighborhood of the points where $d\mathscr{O}(x)$ does not vanish.

On the other hand, let us consider a smooth curve Γ: $u = f(R)$ in the space of hodograph H with the tangent vector $\gamma(R) = df/dR$.

Theorem 1: Let the curve Γ of the class C^1 be such that vector $\gamma(R)$ given by condition (3.1) is a characteristic vector. Let also $\lambda(R)$ be the characteristic covector related to $\gamma(R)$ and $\varphi(\cdot)$ be an arbitrary smooth function of one variable.

If the system

$$\begin{aligned} u(x) &= f(R(x)), \\ R(x) &= \varphi(\lambda_\mu(R(x)) \cdot x^\mu) \end{aligned} \quad (3.2)$$

can be resolved with respect to variables u, R [i.e., $u = u(x)$, $R = R(x)$], then the function $u(x)$ is a solution of the homogeneous system (1.1), i.e.,

$$a^{s\mu}_j \frac{\partial u^j}{\partial x^\mu} = 0. \quad (3.3)$$

This solution is called a simple wave (Riemann wave).

Proof: Differentiating (3.2), we get[23]

$$\begin{aligned} R_{,x^\mu} &= \frac{\dot\varphi}{1 - \dot\varphi (d\lambda_\mu/dR) \cdot x^\mu} \lambda_\mu(R), \\ u^j{}_{,x^\mu}(x) &= f^j{}_{,R}\, R_{,x^\mu} = \frac{\dot\varphi}{1 - \dot\varphi (d\lambda_\mu/dR) \cdot x^\mu} \gamma^j \lambda_\mu, \end{aligned} \quad (3.4)$$

where $\dot\varphi(\lambda_\mu(R) \cdot x^\mu)(d\lambda_\mu(R)/dR) \cdot x^\mu \neq 1$, which is just the condition of a local resolvability of the system (3.3). Then $u^j{}_{,x^\mu}(x)$ is a simple element of the system (3.3) at the point $u(x)$

of the form:

$$du(x) = \xi\gamma \otimes \lambda, \quad \text{i.e.,} \quad \frac{\partial u^j}{\partial x^\mu} = \xi(x)\,\gamma^j(u,\lambda)\cdot\lambda_\mu(u), \qquad (3.5)$$

where the quantity $\xi \not\equiv 0$ is considered as a variable depending on x. Thus, by virtue of the relation (2.1), the function of the form $u^j = f^j(R(x))$ is a solution of Eqs. (1.1). It is well known[22] that the Pfaff system (3.5) is a completely integrable one. Q.E.D.

The form of the solution (3.2) suggests that a covector λ should be treated as an analog of the wave vector $(\omega,\mathbf{k})$, which determines the velocity and the direction of the propagation of the wave. However, unlike the case of linear p.d.e., here $(\omega,\mathbf{k})$ depends also on the value of the solution; therefore, the profile φ of the wave is changing during its propagation.

IV. SIMPLE STATES

In Refs. 1 and 2 it was proved that there exist solutions of the system (1.1) for which the derivative $du(x)$ of the mapping u are the nonhomogeneous simple elements:

$$du(x) = \gamma_0 \otimes \lambda^0, \quad \text{where} \quad a_j^{s\mu}\gamma_0^j\lambda^0_\mu = b^s, \qquad (4.1)$$
$$\lambda^0 = \lambda^0_\mu(u)\,dx^\mu \in \mathscr{E}^*, \quad \gamma_0 = \gamma_0(u,\lambda^0) \in T_uH.$$

In contrast to the condition (3.5) defining the simple wave in the homogeneous system, the expression (4.1) does not include a function ξ of the variable x, and, moreover, the conditions, of integrability are not automatically satisfied. The Frobenius theorem imposes certain conditions, so-called conditions of involutivity, on the form of the covector λ^0 in the formula (4.1). The conditions of compatibility (that is, the symmetry of the second derivatives—Schwarz lemma) are of the form $d(du(x)) = d\gamma_0 \wedge \lambda^0 + \gamma_0 \otimes d\lambda^0 = 0$, modulo Eqs. (3.6), where[23]

$$d\gamma_0 = \gamma_{0,u^i}du^i = \gamma_{0,\gamma_0} \otimes \lambda^0,$$
$$d\lambda^0 = du^i \wedge \lambda^0{}_{,u^i} = \lambda^0 \wedge \lambda^0{}_{,\gamma_0}.$$

Namely, the system (4.1) has a solution (is completely integrable) if $\lambda^0{}_{,\gamma_0} \wedge \lambda^0 = 0$.

It means that the direction of λ^0 is constant along the field γ_0, and then by renormalization $\lambda^0 \to a\lambda^0$, $\gamma_0 \to (1/a)\,\gamma_0$, where $a = a(u)$, one can get λ^0 constant along the field γ_0, i.e.,

$$\lambda^0{}_{,\gamma_0} = 0. \qquad (4.2)$$

Thus the image of a solution is a curve $u = f(R)$ tangent to the field γ_0; what is more, one can choose its parametrization such that $df/dR = \gamma_0(f(R))$. As $\lambda^0(f(R)) = \text{const} =: \hat{\lambda}^0$, so

$$u^j = f^j(\hat{\lambda}^0_\mu \cdot x^\mu) \qquad (4.3)$$

is a solution of a nonhomogeneous equation:

$$a_j^{s\mu}u^j{}_{,x^\mu} = a_j^{s\mu}\frac{df^j}{dR}\lambda^0_\mu = a_j^{s\mu}\gamma_0^j\lambda^0_\mu = b^s.$$

So as a Riemann invariant we can choose the linear function $R(x) = \hat{\lambda}^0_\mu x^\mu$.

Equation (4.3) determines the solution $u = u(x)$ corresponding to the integral curve $u = f(R)$ of the field γ_0. Such solutions will be called simple states.

Let us notice that since $\hat{\lambda}^0$ is a constant 1-form for a simple state so obtained, the solution is constant on $(n-1)$-dimensional hyperplanes, which are parallel to each other and perpendicular to the field $\hat{\lambda}^0$. Hence in this case the gradient catastrophy does not take place.

The name of simple wave we reserve exclusively for solutions of the homogeneous systems. One of the arguments for such an approach is the fact that the accepted definition of the simple state can be used for elliptic equations as well. Such equations have no solutions, which can be interpreted physically as waves. On the other hand it should be noticed that some phenomena which are defined as waves in the traditional (physical) terminology can be described in terms of the notion of the simple state. For example, a localized disturbance propagating in a nonhomogeneous system and called a solitary wave (or a soliton in some cases) corresponds to a simple state. Physically a simple state describes a disturbance which possesses, in contrast to a simple wave, a well-defined profile as well as a constant velocity and direction of propagation. Moreover, the simple state has no gradient catastrophy (i.e., all partial derivatives $|u^j{}_{,x^\mu}|$ of the solution are bounded in the whole domain).

The demand of complete integrability of the system (4.1) is a strong assumption [i.e., system (4.2) may have solutions being not integrable]. So one cannot claim that the method described above gives all simple states of the system. For example, when $m = l = 1$, i.e., for only 1-scalar equation: $a^\mu(u)\,\partial u/\partial x^\mu = b(u)$, each solution is a simple state because $\operatorname{rank}\|\partial u/\partial x^\mu\| = 1$, but most of solutions does not have the form (4.3).

The same may happen in a general case $n > m$, i.e., when the system (1.1) is overdetermined since the solutions being simple states generally do not have to be of the form (4.3).

Let us notice, however, that if

$$u(x) = f(R(x)) \qquad (4.4)$$

is the simple state for system (1.1), for which the matrix $a_j^{s\mu}(f(R))\dot{f}^j(R)$ is a reversible one, then by inserting (4.4) into Eq. (1.1),

$$a_j^{s\mu}(f(R))\dot{f}^j(R)\frac{\partial R}{\partial x^\mu} = b^s(f(R)),$$

we obtain relation of the form $\partial R/\partial x^\mu = \lambda_\mu(R)$. It implies [considering $d(dR) = 0$] that

$$\lambda_{,R} \wedge \lambda = 0. \qquad (4.5)$$

Hence $\lambda(R) = \sigma(R)\,\hat{\lambda}^0$ and thus $R(x) = \Psi(\hat{\lambda}^0_\mu x^\mu)$. Changing the parametrization $R = \Psi(\tilde{R})$, we have $u(x) = \tilde{f}(\tilde{R}(x))$, where $\tilde{R}(x) = \hat{\lambda}^0_\mu x^\mu$ is a linear function.

We now try to find the simple state $u = f(R(x))$ of the nonhomogeneous system (1.1) demanding that the superposition of the functions $x \mapsto R(x)$ and $R \mapsto f(R)$ satisfy automatically the equation

$$a_j^{s\mu}(f(R(x)))\dot{f}^j(R(x))\frac{\partial R}{\partial x^\mu} = b^s(f(R(x))).$$

If we want the condition for variable $R(x)$ to not contain the function $f(\cdot)$, it will be natural to postulate that the function $R(x)$ is the solution of the system $\partial R/\partial x^\mu = \lambda_\mu(R)$. As we know, the function $\lambda_\mu(R)$ must take form $\lambda_\mu(R) = \xi(R)\,\hat{\lambda}^0_\mu$ [vide (4.5)] if the above system is a solvable one.

Let us consider a nonhomogeneous system of Eqs. (1.1)

with $m = 1$. Let us also assume that $\lambda^0 = (\lambda^0_\mu) \in \mathscr{E}$ is such a constant covector that the matrix $\| a^{s\mu}_j(u)\lambda^0_\mu \| := \| c^s_j \|$ is a nonsingular one in the neighborhood of point $u_0 \in H$. Let matrix $\| c^j_s(u) \|$ be the inverse matrix.

Theorem 2. Let us assume that an arbitrary function $\xi(R) \neq 0$ is of a class C^1 and:

(1) $\mathbb{R}^1 \ni R \mapsto f(R) \in H$ is an integral curve of the system of ordinary differential equations: $\dot{f} = (1/\xi)\, c^j_s(f)\, b^s(f)$ satisfying an initial condition $f(0) = u_0$.

(2) $\mathbb{R}^1 \ni \mathscr{O} \mapsto R(\mathscr{O}) \in R^1$ is a solution of the ordinary equation $\dot{R} = \xi(R)$ with an initial condition $R(0) = 0$.

Then the function

$$u(x) := f(R(\lambda^0_\mu x^\mu)) \tag{4.6}$$

is the simple state of system (1.1) satisfying the initial condition $u(0) = u_0$.

Proof results directly from inserting (4.6) into (1.1) and using assumptions (1) and (2):

$$\begin{aligned} a^{s\mu}_j(u)\frac{\partial u^j}{\partial x^\mu} &= a^{s\mu}_j(f(R))\dot{f}^j(R)\cdot\lambda^0_\mu \dot{R}(\lambda^0_\mu x^\mu) \\ &= c^s_j(f(R))\dot{f}^j(R)\,\xi(R) \\ &= [1/\xi(R)]\, b^s(f(R))\cdot\xi(R) = b^s(f(R)) = b^s(u). \end{aligned}$$

Let us notice that the following function $\mathscr{E} \ni x \mapsto R(\lambda^0_\mu x^\mu) \in \mathbb{R}^1$ is Riemann invariant in this case. Q.E.D.

Theorem 3: Let $u = f(R)$ be the curve of class C^1 in the space H and $\lambda^0 = (\lambda^0_\mu) \in \mathscr{E}^*$ be a constant covector; let $\xi(R) \in \mathbb{R}^1$ also be variable such that

$$\xi(R)\, a^{s\mu}_j(f(R))\dot{f}^j(R)\,\lambda^0_\mu = b^s(f(R)). \tag{4.7}$$

If $\varphi(\mathscr{O})$ is the solution of ordinary differential equation of the form $d\varphi/d\mathscr{O} = \xi(\varphi)$, where $\mathscr{O} = \lambda^0_\mu x^\mu$, then

$$u(x) = f(R(x)), \quad R(x) = \varphi(\lambda^0_\mu x^\mu) \tag{4.8}$$

is the simple state.

Proof: In fact, differentiating (4.8), we have

$$\begin{aligned} \frac{\partial u^j}{\partial x^\mu} &= \dot{f}^j(R)\frac{d\varphi}{d\mathscr{O}}\lambda^0_\mu \\ &= \dot{f}^j(R)\,\xi(R)\,\lambda^0_\mu; \end{aligned}$$

hence

$$\begin{aligned} a^{s\mu}_j\frac{\partial u^j}{\partial x^\mu} &= a^{s\mu}_j(f(R))\dot{f}^j(R)\,\xi(R)\,\lambda^0_\mu \\ &= b^s(f(R)). \end{aligned} \tag{4.9}$$

Q.E.D.

Let us denote

$$c(R) := \| a^{s\mu}_j(f(R))\dot{f}^j(R) \| \in L(\mathscr{E}^*, \mathbb{R}^m).$$

Then the one-parameter family of subspaces

$$c(R)^{-1}\cdot\{\operatorname{span} b(f(R)) \setminus \{0\}\} \quad (\text{when } m = n)$$

must have a common element in $\mathscr{E}^*$—i.e., (λ^0_μ). This is the condition guaranteeing the existence of $\lambda \in \mathscr{E}^*$ and $\xi(R) \in \mathbb{R}^1$.

Theorem 4: Let $R \mapsto f(R)$ be a curve of class C^1 in the hodograph space H. Let us assume that the linear homogeneous system of equations

$$B^{s\mu}(R)\frac{\partial F}{\partial x^\mu} + B^s(R)\frac{\partial F}{\partial R} = 0, \tag{4.10}$$

where $B^{s\mu}(R) := a^{s\mu}_j(f(R))\dot{f}^j(R)$, and $B^s(R) := b^s(f(R))$ has a solution

$$F = F(R, x).$$

If the system

$$u = f(R), \quad F(R, x) = 0 \tag{4.11}$$

is uniquely resolved with respect to R, u then the function $u(x)$ is a simple state of nonhomogeneous system (1.1).

Proof: From resolvability (4.11) it arises that one can resolve the equation $F(R, x) = 0$ with respect to R, if

$$\frac{\partial R}{\partial x^\mu} = -\frac{\partial F/\partial x^\mu}{\partial F/\partial R}.$$

From (4.10) it results

$$B^{s\mu}(R)\frac{\partial R}{\partial x^\mu} = B^s(R); \tag{4.12}$$

hence $a^{s\mu}_j(f(R))\dot{f}^j(R)\,\partial R/\partial x^\mu = b^s(f(R))$.

That ends the proof, since in presence of Eq. (4.11) we have $\partial u^j/\partial x^\mu = \dot{f}^j(R)\,\partial R/\partial x^\mu$. Q.E.D.

Remark 1: If $m = n$ and the matrix $\| B^{s\mu}(R) \|$ is a reversible one, then Eq. (4.12) can be written in the form

$$\frac{\partial R}{\partial x^\mu} = \lambda_\mu(R),$$

where $\lambda_\mu := B^s b_{s\mu}$ and $\| b_{s\mu} \| := \| B^{s\mu} \|^{-1}$. Hence, as we know [vide (4.4)],

$$\lambda_\mu(R) = \lambda^0_\mu \xi(R), \quad R(x) = R(\lambda^0_\mu x^\mu)$$

and equation

$$B^{s\mu}(R)\lambda_\mu(R) = B^s(R)$$

gives exactly Eq. (4.6). Thus in this case both methods are equivalent.

V. THE EXAMPLES OF APPLICATIONS OF SIMPLE STATES FOR THE NONLINEAR NONHOMOGENEOUS EQUATIONS

Now we shall present a few examples which will illustrate the theoretical considerations. Let us consider the equation of the second order:

$$g^{\mu\nu}\varphi_{,x^\mu x^\nu} = b(\varphi), \quad \mu,\nu = 0,1,2,3 \tag{5.1}$$

(where $g^{\mu\nu}$ is a constant metric tensor with arbitrary signature—Riemann or Lorentz, for example), which can be used for the description of the Josephson phenomenon in superconductivity, in Euclidean field theory, in the theory of elementary particles, in electrodynamics, in magnetohydrodynamics, in gas dynamics, etc.

According to the requirements of the method, we reduce Eq. (5.1) to the system of equations of the first order:

$$\begin{aligned} &g^{\mu\nu}\varphi_{\mu,x^\nu} = b(\varphi), \quad \varphi_{,x^\mu} = \varphi_\mu, \\ &\varphi_{\mu,x^\nu} - \varphi_{\nu,x^\mu} = 0, \quad \mu,\nu = 0,1,2,3. \end{aligned} \tag{5.2}$$

We have gotten the system of 11 equations for five unknown functions $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi)$. Introducing real simple integral

elements, we get

$$g^{\mu\nu}\gamma^{\mu}\lambda_{\nu} = b(\varphi), \quad \gamma^{4}\lambda_{\mu} = \varphi_{\mu},$$
$$\gamma^{\mu}\lambda_{\nu} - \gamma^{\nu}\lambda_{\mu} = 0, \quad \mu,\nu = 0,1,2,3, \tag{5.3}$$

where $d\varphi_{\mu}/dR = \gamma^{\mu}$ and $d\varphi/dR = \gamma^{4}$. Thus the simple integral nonhomogeneous elements have the form

$$\gamma = \left(\frac{b(\varphi)}{g^{\mu\nu}\lambda_{\mu}\lambda_{\nu}}(\lambda_0, \boldsymbol{\lambda}), \frac{\varphi_0}{\lambda_0}\right),$$
$$\frac{\varphi_0}{\lambda_0} = \frac{\varphi_1}{\lambda_1} = \frac{\varphi_2}{\lambda_2} = \frac{\varphi_3}{\lambda_3},$$
$$\lambda = (\lambda_0, \boldsymbol{\lambda}), \quad \boldsymbol{\lambda} := (\lambda_1, \lambda_2, \lambda_3). \tag{5.4}$$

The solution—simple state—is given by the expression (4.3). The problem is reduced then to solving a system of ordinary differential equations:

$$\frac{d\varphi_{\mu}}{dR} = \frac{b(\varphi)}{g^{\rho\nu}\lambda_{\rho}\lambda_{\nu}}\lambda_{\mu},$$
$$\frac{d\varphi}{dR} = \frac{\varphi_0}{\lambda_0} = \frac{\varphi_1}{\lambda_1} = \frac{\varphi_2}{\lambda_2} = \frac{\varphi_3}{\lambda_3}, \tag{5.5}$$

where $R = \lambda_{\mu} \cdot X^{\mu}$ and λ_{μ} are arbitrary real constants. Hence we have

$$g^{\mu\nu}\lambda_{\mu}\lambda_{\nu}(\dot{\varphi})^2 = 2\left[\int b(\varphi)\,d\varphi + E\right], \tag{5.6}$$

where E is an arbitrary real integration constant.

A. The field equation

Now let us consider a particular wave equation (5.1) in the case when its right-hand side has a form $b(\varphi) = 4A\varphi^3 + 3B\varphi^2 + 2C\varphi + D$, namely,

$$g^{\mu\nu}\varphi_{,x^{\mu}x^{\nu}} = 4A\varphi^3 + 3B\varphi^2 + 2C\varphi + D, \quad A,B,C,D \in \mathbb{R}^1.$$

The solution—simple state—is given here by virtue of Eq. (5.6) by the following expression:

$$\tfrac{1}{2}g(\dot{\varphi})^2 = A\varphi^4 + B\varphi^3 + C\varphi^2 + D\varphi + E,$$
$$g := g^{\mu\nu}\lambda_{\mu}\lambda_{\nu}. \tag{5.7}$$

The general solution of this equation is given by the elliptic function.[24] Thus we have

$$\varphi(R) = \sqrt{g/2A}\, f(R,t), \quad g/2A > 0,$$

where the function f is given by the formulas

$$f(R,t) = \frac{1}{2}\frac{P'(R + \tfrac{1}{2}t, \omega_1,\omega_2) + P'(R - \tfrac{1}{2}t, \omega_1,\omega_2)}{P(R + \tfrac{1}{2}t, \omega_1,\omega_2) - P(R - \tfrac{1}{2}t, \omega_1,\omega_2)} - a_1,$$
$$R = \lambda_{\mu}\, x^{\mu},$$

and the quantity t satisfies the conditions

$$P(t, \omega_1,\omega_2) = a_1^2 - a_2,$$
$$P'(t, \omega_1,\omega_2) = 2a_1^3 + a_3 - 3a_1a_2, \tag{5.8}$$

where $a_1 := B/2\sqrt{2Ag}$, $a_2 := C/3g$, $a_3 := (D/g)\sqrt{A/2g}$, $a_4 = 4AE/g^2$ and where P is the Weierstrass P-function[24] satisfying

$$P'^2 = 4P^3 - g_2P - g_3. \tag{5.9}$$

The so-called invariants g_2 and g_3 are homogeneous functions of the periods ω_1 and ω_2 of the fourth and sixth order, respectively, and are given by formulas

$$g_2(\omega_1, \omega_2) = 60 \sum_{m,m'}{}' \frac{1}{(m\omega_1 + m'\omega_2)^4},$$
$$g_3(\omega_1, \omega_2) = 140 \sum_{m,m'}{}' \frac{1}{(m\omega_1 + m'\omega_2)^6}.$$

In our case the invariants g_2 and g_3 are given by formulas

$$g_2(\omega_1, \omega_2) = a_4 - 4a_1a_3 + 3a_2^2 =: q,$$
$$g_3(\omega_1, \omega_2) = \det\begin{vmatrix} 1 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} =: p. \tag{5.10}$$

The existence of the periods ω_1 and ω_2 guarantees the following fact.[25] For every real number c the equation $J(\tau) = c$ [where $J(\tau) = g_2^3/(g_2^3 - 27g_3^2)$ is the module function of $\tau := \omega_1/\omega_2$] possesses exactly one root in the fundamental region of the modulator group. Thus, taking $c := q^2/(q^3 - 27p^2)$, we can obtain the ratio $\tau = \omega_1/\omega_2$. If $g_2 = g \neq 0$, then from the homogeneity of the function g_2 we can determine

$$\omega_1^4 = q^{-1}\, g_2(1,\tau)$$

and, when $g_2 = q = 0$, we have

$$\omega_1^6 = p^{-1}\, g_3(1,\tau).$$

When ω_1 is found, then ω_2 is determined from the formula $\omega_2 = \tau^{-1}\,\omega_1$. The periods ω_1 and ω_2 calculated in such a way satisfy (5.10). One knows[25] that such t satisfying Eqs. (5.8) always exists.

By specialization of the right-hand side of Eq. (5.7) we get particular equations appearing in various branches of mathematical physics.

For example, let us consider the equations of motion for the massive SU(2) Yang–Mills theory

$$\frac{\partial G^{a}_{\mu\nu}}{\partial x^{\nu}} = e\epsilon_{abc}\, G^{b}_{\mu\nu}w^{c}_{\nu} + \mu^2\, w^{a}_{\mu},$$

where

$$G^{a}_{\mu\nu} = \frac{\partial w^{a}_{\nu}}{\partial x^{\mu}} - \frac{\partial w^{a}_{\mu}}{\partial x^{\nu}} + a\epsilon_{abc}\, w^{b}_{\mu}\, w^{c}_{\nu}.$$

By the so-called t'Hooft–Coorigan–Fairlie–Wilczek ansatz

$$ew^{a}_{0} = i\frac{\partial\varphi}{\partial x^{a}}\Big/\varphi,$$
$$ew^{a}_{i} = \epsilon_{ian}\frac{\partial\varphi}{\partial x^{n}}\Big/\varphi + i\delta_{ai}\frac{\partial\varphi}{\partial x^{0}}\Big/\varphi,$$

the potential w^{a}_{μ} may be reduced to a potential φ satisfying the scalar Φ^4-theory equation

$$\Box\varphi - \tfrac{1}{2}\mu^2\varphi + \lambda\varphi^3 = 0,$$
$$\Box := \frac{\partial^2}{\partial(x^0)^2} - \sum_{s=1}^{3}\frac{\partial^2}{\partial(x^s)^2}, \tag{5.11}$$

with the initial conditions

$$\varphi(0)=\varphi_0,\quad \frac{\partial\varphi(0)}{\partial x^0}=0.$$

In our case the ordinary equation (5.7) can be reduced by a particular choice of constants $A=-\lambda/4$, $C=\mu^2/4$, $B=D=0$ to the elliptic equation

$$2g(\dot{\varphi})^2=\mu^2\varphi^2-\lambda\varphi^4+4E,\quad \text{where } R=\lambda_\mu x^\mu,$$

which can be solved by Jacobi functions.[24] Thus we have [26,27]

$$\varphi=\varphi_0\,\mathrm{sn}(A_1\lambda_\mu x^\mu+E_1,k_1),\quad \mathrm{sn}\,E_1=1,$$
$$\varphi=\varphi_0\,\mathrm{cn}(A_1\lambda_\mu x^\mu+E_2,k_2),\quad \mathrm{cn}\,E_2=1,$$
$$\varphi=\varphi_0\,\mathrm{dn}(A_3\lambda_\mu x^\mu+E_3,k_3),\quad \mathrm{dn}\,E_3=1,$$

and

$$A_1^2=a+\tfrac{1}{2}b\varphi_0^2,\quad k_1^2=-\lambda\varphi_0^2/2A_1^2 g,\quad a:=-\mu^2/2g,$$
$$A_2^2=a+b\varphi_0^2,\quad k_2^2=\lambda\varphi_0^2/4A_2^2 g,\quad b:=\lambda/g,$$
$$A_3^2=b\varphi_0^2/2,\quad k_3^2=k_1^{-2},\quad g:=\lambda_0^2-\lambda^2.$$

where k_i are the Jacobi parameters. In this particular case the solutions can be interpreted as "periodic waves" (periodic potentials).

At present let us consider a particular d'Alembert equation of the form

$$\Box\varphi=\mu\varphi+\varphi^2,\quad \mu\in\mathbb{R}^1. \tag{5.12}$$

Using the general procedure presented in this paper, one gets an ordinary differential equation which turns out to be identical with Eq. (5.7) with the particular choice of constants $B=\frac{1}{2}$, $C=\mu$, $A=D=0$.

The general solution of this equation is again the Weierstrass P-function, satisfying Eq. (5.9). In this case the invariants g_2 and g_3 are given by the formulas:

$$g_2(\omega_1,\omega_2)=\frac{\mu^2}{12g^2},$$
$$g_3(\omega_1,\omega_2)=-\left(\frac{\mu}{6g}\right)^3+\frac{E}{g}\frac{1}{(6g)^2}.$$

We can determine the periods of Weierstrass P-functions ω_1 and ω_2. Then the solution of Eq. (5.12) takes the form

$$\varphi=6gP(R,\omega_1,\omega_2)-\tfrac{1}{2}\mu,\quad R=\lambda_\mu x^\mu.$$

Now let us consider the asymptotic situation where one of the periods of the Weierstrass P-function becomes infinite, for example, $\omega_2=\infty$. In this case we can express the solution with the help of the trigonometric functions. The results are easily derivable.

B. The Liouville equation

Now let us start with Liouville equation in four dimensions:

$$\Box\varphi=\mu\exp\varphi,\quad \mu\in\mathbb{R}^1. \tag{5.13}$$

In one-dimensional case the general solution is well known[28]:

$$\varphi=\ln\left(\frac{8}{|\mu|}\frac{f'(x+t)\,g'(x-t)}{[f(x+t)-g(x-t)]^2}\right),$$

where f and g are arbitrary functions of one variable and f' and g' are their derivatives.

In a more general case of four dimensions, our procedure gives for a solution [the simple state of Eq. (5.13)] the following expression:

$$\varphi=\ln\left[P(\pm\sqrt{\mu/2(\lambda_0^2-\lambda^2)}\,\lambda_\mu x^\mu,\omega_1\omega_2)-(1/3\mu)D\right], \tag{5.14}$$

where P is the Weierstrass P-function satisfying Eq. (5.9) with the invariants g_2 and g_3 given in the following form:

$$g_2(\omega_1,\omega_2)=(4/3\mu^2)D^2,\quad g_3(\omega_1,\omega_2)=(-8/27\mu^3)D^3. \tag{5.15}$$

Equation (5.13) has the physical interpretation in the Euclidean quantum field theory. According to Ref. 29 particles are described as the singular solutions of the Liouville equation. In our case, solution (5.14) with the conditions (5.15) describes the motion of a single particle in the Euclidean quantum field theory.

Equation (5.13) also has another interpretation in plasma phsyics. Let us consider electrical potential v created by particle distribution (ions, electrons, etc. with charge Ze, where e = elementary charge and Z = nonzero integer number) at absolute temperature T. Let the concentration of charge particles be n_0. Then we have the following equation for potential[30]:

$$\Delta v=-4\pi e n_0\exp(-Zev/kT).$$

By substituting $\varphi=-Zev/kT$ we obtain Eq. (5.13), where $\mu=4\pi Ze^2n_0/kT$. Thus the obtained solution (5.14) with the condition (5.15) describes the self-consistent potential created by charged particles at temperature T. This potential always has a singular point, given by

$$P(\pm\sqrt{\mu/2(\lambda_0^2-\lambda^2)}\,\lambda_\mu x^\mu,\omega_1,\omega_2)-(1/3\mu)D=0.$$

Therefore, we can consider this solution as a logarithmic potential created by an effective point charge located at a singular point.

C. The sine–Gordon equation

Let us consider now the four-dimensional sine–Gordon equation:

$$\Box\varphi=\sin\varphi \tag{5.16}$$

with the initial conditions

$$\varphi(0)=\varphi_0,\quad \frac{\partial\varphi(0)}{\partial x^0}=0. \tag{5.17}$$

In this case the solutions of Eq. (5.16a)—the simple states—are given by the expressions:

$$\varphi=4\arctan[\varphi_0\,\mathrm{sn}(A_1\lambda_\mu x^\mu+E_1,k_1)]+(2j+1)\pi,\quad \mathrm{sn}\,E_1=1,$$
$$\varphi=4\arctan[\varphi_0\,\mathrm{cn}(A_2\lambda_\mu x^\mu+E_2,k_2)]+(2j+1)\pi,\quad \mathrm{cn}\,E_2=1, \tag{5.18}$$
$$\varphi=4\arctan[\varphi_0\,\mathrm{dn}(A_3\lambda_\mu x^\mu+E_3,k_3)]+(2j+1)\pi,\quad \mathrm{dn}\,E_3=1.$$

and

$$A_1^2 = \frac{-(D+3)}{4g} - \frac{(D-1)}{8g}\varphi_0^2, \quad k_1^2 = \frac{(D-1)}{8g}\frac{\varphi_0^2}{A_1^2},$$
$$A_2^2 = \frac{-(D+3)}{4g} - \frac{(D-1)}{4g}\varphi_0^2, \quad k_2^2 = \frac{(1-D)}{8g}\frac{\varphi_0^2}{2A_2^2}, \tag{5.19}$$
$$A_3^2 = \frac{(1-D)}{8g}\varphi_0^2, \quad k_3^2 = k_1^{-2}, \quad \varphi_0^4 + 2(D+3)\varphi_0^2 - 1 = 0,$$

where $g := \lambda_0^2 - \lambda^2$, $D \in \mathbb{R}^1$, $j \in Z$.

D. The cosh–and sinh–d'Alembert equations

Let us consider the semilinear system of equations of the form:

$$\text{(a)}\ \Box\varphi = \cosh\varphi, \quad \text{(b)}\ \Box\varphi = \sinh\varphi. \tag{5.20}$$

The solution of Eq. (5.20)—the simple state—is given by the expression

$$\varphi = \ln\left[4gP(R, \omega_1, \omega_2) - \tfrac{2}{3}D\right], \tag{5.21}$$
$$R = \lambda_\mu x^\mu, \quad D \in \mathbb{R}^1, \quad g := \lambda_0^2 - \lambda^2,$$

where P is the Weierstrass P-function satisfying Eq. (5.9) with the invariants g_2 and g_3 given, respectively, by the formulas

$$g_2(w_1, \omega_2) = (1/4g^2)(\tfrac{4}{3}D^2 \pm 1),$$
$$g_3(\omega_1, \omega_2) = (-D/g^3)(D^2/27 \pm \tfrac{1}{24}), \tag{5.22}$$

where the upper sign refers to case (a) and the lower to case (b).

E. The Korteweg–de Vries equation

Finally let us consider the Korteweg de Vries equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{5.23}$$

Reducing Eq. (5.23) to the system of the first-order equations of the form

$$u_t + p_x = -6uv, \quad u_x = v, \quad v_x = p, \tag{5.24}$$

we get a system of three equations for the three unknown functions (p,v,u). Introducing the notion of real simple integral elements we get

$$\gamma^3\lambda_1 + \gamma^1\lambda_2 = -6uv, \quad \gamma^2\lambda_2 = p, \quad \gamma^3\lambda_2 = v, \tag{5.25}$$

where $dp/dR = \gamma^1$, $dv/dR = \gamma^2$, $du/dR = \gamma^3$. Thus the simple integral nonhomogeneous elements have the form of

$$\gamma = (1/\lambda_2)(-v(6u + \lambda_1/\lambda_2), p, v), \quad \lambda = (\lambda_1, \lambda_2).$$

The simple state, according to the discussion in Sec. IV. can here be made to satisfy the following conditions:

$$\frac{dp}{dR} = \frac{-v}{\lambda_2}\left(6u + \frac{\lambda_1}{\lambda_2}\right), \quad \text{where } R = \lambda_1 t + \lambda_2 x,$$
$$\frac{du}{dR} = \frac{v}{\lambda_2}, \quad \frac{dv}{dR} = \frac{p}{\lambda_2}, \tag{5.26}$$

where λ_1 and λ_2 are arbitrary real constants. Then we may reduce Eq. (5.26) to the ordinary differential equation for the variable u in the form:

$$\left(\frac{du}{dR}\right)^2 = \frac{1}{\lambda_2^2}\left(c_2 + 2c_1u - u^2\frac{\lambda_1}{\lambda_2} - 2u^3\right), \tag{5.27}$$

where c_1 and c_2 are arbitrary real constants. If we substitute $u = -(y + \lambda_1/6\lambda_2)$ and exchange variable $s = (3/2\lambda_2)^{1/2}R$, we can reduce Eq. (5.27) to the equivalent form:

$$\left(\frac{dy}{ds}\right)^2 = 4y^3 - \frac{2}{3}\left[2c_1 + \frac{1}{6}\left(\frac{\lambda_1}{\lambda_2}\right)^2\right]$$
$$- c_2 + \frac{c_1\lambda_1}{3\lambda_2} + \frac{1}{54}\left(\frac{\lambda_1}{\lambda_2}\right)^2. \tag{5.28}$$

The general solution of this equation is again the Weierstrass P-function satisfying Eq. (5.9) with the invariants g_2 and g_3 given by the formulas:

$$g_2 = \frac{2}{3}\left[2c_1 + \frac{1}{6}\left(\frac{\lambda_1}{\lambda_2}\right)^2\right],$$
$$g_3 = c_2 - \frac{c_1\lambda_1}{3\lambda_2} - \frac{1}{54}\left(\frac{\lambda_1}{\lambda_2}\right)^2.$$

Hence we can write the solution of the KdV equation in the following form[31]:

$$u(t,x) = -P(\lambda_1 t + \lambda_2 x + D, \omega_1, \omega_2)$$
$$-\lambda_1/6\lambda_2, \quad D \in \mathbb{R}^1, \tag{5.29}$$

which is a particular case of the so-called solitary wave and it is a one-soliton solution.

Finally it should be noted that the Weierstrass P-functions appear in all solutions of the nonhomogeneous wave equations of the form (5.1) as well as in the Korteweg–de Vries equation. It seems that the known soliton equations are closely connected with the elliptical functions, but this fact needs further investigations.

VI. NONLINEAR SUPERPOSITION OF SIMPLE WAVES

There have been also studied[10,12–17] the solutions of the homogeneous systems (3.3) representing the nonlinear superposition of simple waves, i.e., the double wave and in general the k waves. A form of solutions of the system (3.3) for which the derivative $du(x)$ of the mapping u is the sum of homogeneous simple elements has been proposed

$$du(x) = \sum_{r=1}^{k} \xi^r \gamma_r \otimes \lambda^r,$$
$$\lambda^{r_1} \wedge \lambda^{r_2} \wedge \lambda^{r_3} \neq 0 \quad \text{for } r_1 < r_2 < r_3, \ k \leqslant 1, \tag{6.1}$$
$$a_j^{s\mu}\gamma_{(r)}^j \lambda_\mu^r = 0,$$

where quantities $\xi^r \neq 0$ are treated as variables dependent on x. It was proved[10,12] that the solutions of the postulated form (6.1) exist and can be written in Riemann invariants. It means that in the latter case (Riemann invariants) we make the additional assumption that the commutators for all vector fields γ_r and γ_s' are the linear combinations of these fields, i.e.,

$$\bigwedge_{r,s\in\{1,\ldots,k\}} [\gamma_r, \gamma_s] \in \operatorname{span}\{\gamma_r, \gamma_s\}.$$

So if these conditions are satisfied, then we may change the length of the vectors γ_r and γ_s such that $[\gamma_r, \gamma_s] = 0$ and the vectors $\gamma_1,\ldots,\gamma_k$ constitute a holonomic system, and there exists a parametrization of the surface tangent to the field $\gamma_1,\ldots,\gamma_k$:

$$u = f(R), \quad \text{where } R := (R^1,\ldots,R^k) \in \mathbb{R}^k, \tag{6.2}$$

such that

$$\frac{\partial f(R)}{\partial R^r} \sim \gamma_r(R) \in T_{f(R)} H. \tag{6.3}$$

Consequently,

$$du(x) = \sum_{r=1}^{k} \frac{\partial f}{\partial R^r} dR^r,$$

$$dR^r = \sum_{\mu=1}^{n} \frac{\partial R^r}{\partial x^\mu} dx^\mu,$$

which together with the assumption that $\gamma_1,\ldots,\gamma_k$ are linearly independent, leads to a system of Pfaff forms

$$dR^r(x) = \xi^r \lambda^r(R(x)), \quad \text{where } \xi^r(x) \neq 0, \ r \in \{1,\ldots,k\}. \tag{6.4}$$

The fields of covectors λ^r become functions of the parameter R (i.e., $\lambda^r = \sum_{\mu=1}^n \lambda^r_\mu(R)\, dx^\mu \in \mathscr{E}^*$). Involutivity conditions for the system (6.4) were already investigated.[10,12] Namely the system (6.4) has solutions (is integrable) if the following conditions are satisfied[12]:

$$\bigwedge_{r\neq s\in\{1,\ldots,k\}} \frac{\partial \lambda^s}{\partial R^r} = \alpha^s_r \lambda^s + \beta^s_r \lambda^r. \tag{6.5}$$

These relations are necessary and sufficient conditions for the existence of the solutions of the system (6.4). They ensure us that the set of solutions of the system (6.4) depends on k arbitrary functions of one variables. From Eqs. (6.5) we have

$$\frac{\partial^2 \lambda^s}{\partial R^r \partial R^t} = \left(\alpha^s_r \alpha^s_t + \frac{\partial \alpha^s_r}{\partial R^t}\right)\lambda^s + \left(\beta^s_r \alpha^r_t + \frac{\partial \beta^s_r}{\partial R^t}\right)\lambda^r + (\beta^s_r \beta^r_t + \alpha^s_r \beta^s_t)\lambda^t, \quad r \neq t \neq s.$$

Hence the compatibility conditions (that is, the symmetry of the second derivatives) are of the form

$$\frac{\partial \beta^s_r}{\partial R^t} = \beta^s_t \beta^t_r - \beta^s_r \alpha^r_t + \beta^s_r \alpha^s_t \quad \text{for } r \neq s \neq t, \tag{6.6}$$

$$\frac{\partial \alpha^s_r}{\partial R^t} - \frac{\partial \alpha^s_t}{\partial R^t} = 0 \quad \text{for } r,t \neq s. \tag{6.7}$$

Equations (6.7) give

$$\exists\ \Psi^s(R): \quad \alpha^s_r = -\frac{\partial \Psi^s}{\partial R^r} \quad \text{for } r \neq s.$$

Hence $\varphi^s := \exp(\Psi^s)$ satisfies

$$\frac{\partial \varphi^s}{\partial R^r} + \varphi^s \alpha^s_r = 0.$$

Introducing the notation

$$\bar{\lambda}^s := \varphi^s \lambda^s, \tag{6.8}$$

we obtain

$$\frac{\partial \bar{\lambda}^s}{\partial R^r} = \beta^s_r \bar{\lambda}^r \quad \text{for } r \neq s, \tag{6.9}$$

and Eq. (6.6) takes the form

$$\frac{\partial \bar{\beta}^s_r}{\partial R^t} = \bar{\beta}^s_t \bar{\beta}^t_r \quad \text{for } r \neq s \neq t. \tag{6.10}$$

Let us notice that replacing λ^s by $\bar{\lambda}^s$ does not change the basic system of partial differential equation for $R^s(x)$.

For $k = 1$ the compatibility conditions (6.5) are automatically satisfied (do not exist).

Let us study now an example of the formulation of the Cauchy problem for the Pfaff system (6.5) in the case of double waves.[33] The conditions (6.5) can be written in an equivalent form. We can always choose a normalization of the length of covectors $\lambda^s \in \mathscr{E}^*$ such that $\lambda^s = (1, \lambda^s_1,\ldots,\lambda^s_n)$, $s = 1,2$. If $\lambda^1{}_{,R^2}$ and $\lambda^2{}_{,R^1} \in \operatorname{span}\{\lambda^1, \lambda^2\}$, then there exist functions α^1_2 and β^2_1 of $R = (R^1, R^2)$ such that

$$\lambda^1{}_{,R^2} = \alpha^1_2(\lambda^1 - \lambda^2), \tag{6.11a}$$

$$\lambda^2{}_{,R^1} = \beta^2_1(\lambda^2 - \lambda^1). \tag{6.11b}$$

Differentiating (6.11a) with respect to R^1 and utilizing Eq. (6.11b) [or vice versa—differentiating (6.11b) with respect to R^2 and utilizing Eq. (6.11a)], we obtain linear hyperbolic system of n equations of second order for the unknown functions λ^s, i.e.,

$$\lambda^s{}_{,R^1R^2} + a^s \lambda^s{}_{,R^1} + b^s \lambda^s{}_{,R^2} \tag{6.12}$$

where the quantities

$$a^1 := -\alpha^1_2, \quad b^1 := -(\beta^2_1 + \alpha^1_{2,R^1}/\alpha^1_2),$$

$$a^2 := \alpha^1_2 - \beta^2_{1,R^2}/\beta^2_1, \quad b^2 := \beta^2_1$$

are treated as the given functions of R in the domain $\mathscr{B} \subset \mathscr{H}$, having continuous second derivatives in $\mathscr{B}$. Equation (6.12) [just as Eq. (6.11)] separates into n independent scalar equations for the components of the vector λ^s

$$u_{,R^1R^2} + a^s u_{,R^1} + b^s u_{,R^2} = 0, \quad s = 1,2 \tag{6.13}$$

(u stands for a given component of λ^s). Thus we obtain a hyperbolic equation with two independent variables. The classical result concerning the Cauchy problem for the equation

$$u_{,x^1x^2} + a u_{,x^1} + b u_{,x^2} = 0 \tag{6.14}$$

is well known.[34] We specify the value of the function u and an outgoing derivative of u along a noncharacteristic initial curve Γ on the plane (x^1,x^2), i.e., $u|_\Gamma$ and $du/dz|_\Gamma$, are given. The solution of the problem (6.14) exists and is unique in the rectangle bounded by characteristics containing the curve Γ (Fig. 1). It can be expressed by means of the so-called Riemann function v by the formula

$$U(P) = \tfrac{1}{2}[v(A)u(A) + v(B)u(B)] - \int_\Gamma \left[v\frac{du}{dz} - u\frac{dv}{dz} + uv(a\eta_1 + b\eta_2)\right] ds, \tag{6.15}$$

where η_1 and η_2 denote the direction of the outgoing normal.

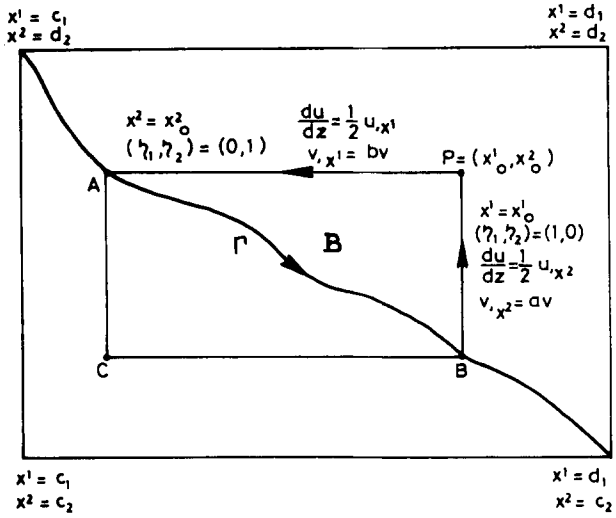

FIG. 1.

By du/dz and dv/dz we denote the directional derivative $\frac{1}{2}(\eta_2 u_{x^1} + \eta_1 u_{x^2})$ and $\frac{1}{2}(\eta_2 v_{x^1} + \eta_1 v_{x^2})$, respectively. Riemann function v for Eq. (6.14) can be defined by the conditions

$$\begin{aligned} &v_{x^1x^2} - av_{x^1} - bv_{x^2} - (a_{x^1} + b_{x^2})v = 0, \\ &v_{,x^2} = av \quad \text{along} \quad x^1 = \text{const} = x_0^1, \\ &v_{,x^1} = bv \quad \text{along} \quad x^2 = \text{const} = x_0^2. \end{aligned} \tag{6.16}$$

The conditions (6.16) are equivalent to

$$v = Tv + 1, \tag{6.17}$$

where T is an integral operator of the form

$$Tv = -\int_C^B (bv)\,dx^1 - \int_C^A (av)\,dx^2.$$

It can be proved[34] that for arbitrary v the following inequality holds:

$$|Tv| \leqslant \tfrac{1}{2}\max|v|. \tag{6.18}$$

From Eq. (6.18) it follows that Eq. (6.17) has exactly one solution (which can be obtained by the iterative scheme $v_{n+1} = Tv_n + 1$ and $v_0 = 1$). This proves that the Riemann function for Eq. (6.14) exists and is unique.

We prove now for an arbitrary $k \geqslant 1$ that all the solutions of the system (6.4) can be obtained [taking into account (6.5)] by resolution with respect to the variable $R^1,\ldots,R^k$ of the system

$$\lambda^s(R)\cdot x = \Psi^s(R), \tag{6.19}$$

where Ψ^s are the arbitrary functions of class C^1 of the variable $R = (R^1,\ldots,R^k)$.

Indeed, differentiation of Eq. (6.19) gives us

$$\left(\frac{\partial\Psi^s}{\partial R^s} - \frac{\partial(\lambda^s x)}{\partial R^s}\right) dR^s - \lambda^s = \sum_{r\neq s=1}^{k} \left(\frac{\partial(\lambda^s x)}{\partial R^r} - \frac{\partial\Psi^s}{\partial R^r}\right) dR^r. \tag{6.20}$$

If the system (6.4) is satisfied, then the left-hand side is proportional to $\lambda^s(R)$ and the right-hand side is a linear combination of $\lambda^r(R)$ for $r \neq s$. Thus, if $\xi^s \neq 0$, then

$$\frac{\partial(\lambda^s x)}{\partial R^r} - \frac{\partial\Psi^s}{\partial R^r} = 0. \tag{6.21}$$

From (6.21) and from the compatibility conditions (6.5), we have

$$\frac{\partial\Psi^s}{\partial R^r} = \alpha_r^s(\lambda^s x) + \beta_r^s(\lambda^r \cdot x)$$

so, by virtue of (6.19), we obtain[35]

$$\frac{\partial\Psi^s}{\partial R^r} = \alpha_r^s\Psi^s + \beta_r^s\Psi^s, \quad r \neq s. \tag{6.22}$$

Then Eq. (6.20) gives

$$\left(\frac{\partial\Psi^s}{\partial R^s} - \frac{\partial(\lambda^s \cdot x)}{\partial R^s}\right) dR^s = \lambda^s,$$

where $\partial\Psi^s/\partial R^s - \partial(\lambda^s x)/\partial R^s \neq 0$ is the resolvability condition. So we have Eq. (6.4).

If for example $\xi^1 \equiv 0$, then $R^1 = R_0^1 = \text{const}$. From the integrability conditions (6.5) for the covectors $\lambda^s(\tilde{R}) := \lambda^s(R_0^1, \tilde{R})$, $s = 2,\ldots,k$, where $\tilde{R} = (R^2,\ldots,R^k)$, we obtain

$$\frac{\partial\tilde{\lambda}^s}{\partial R^r} = \tilde{\alpha}_r^s\tilde{\lambda}^s + \tilde{\beta}_r^s\tilde{\lambda}^r \quad \text{for } s \neq r \in \{2,\ldots,k\},$$

where the functions $\Psi^2,\ldots,\Psi^k$ satisfy the conditions (6.19) and (6.22) for $r \geqslant 2$. Thus the solution is given by $R^1 = R_0^1$ and by the function $R^2,\ldots,R^k$ obtained by the resolution of the system

$$\Psi^s(\tilde{R}) = \lambda_\mu^s(\tilde{R})\,x^\mu \quad \text{for } s \geqslant 2.$$

In particular, for $k = 1$ we obtain the formula (6.19) for the simple wave

$$\Psi(R) = \lambda_\mu(R)x^\mu \Leftrightarrow R = \Psi^{-1}(\lambda_\mu(R)x^\mu).$$

Let us investigate now the compatibility conditions for the system (6.22). It is a system of $k^2 - k$ equations for k unknown functions. We have

$$\begin{aligned} \frac{\partial^2\Psi^s}{\partial R^r\partial R^t} &= \frac{\partial\alpha_r^s}{\partial R^t}\Psi^s + \frac{\partial\beta_r^s}{\partial R^t}\Psi^r + \alpha_r^s(\alpha_t^s\Psi^s + \beta_t^s\Psi^t) \\ &\quad + \beta_t^s(\alpha_t^r\Psi^r + \beta_t^r\Psi^t) \\ &= \left(\frac{\partial\alpha_r^s}{\partial R^t} + \alpha_r^s\alpha_t^s\right)\Psi^s + \left(\frac{\partial\beta_r^s}{\partial R^t} + \beta_r^s\alpha_t^r\right)\Psi^r \\ &\quad + (\alpha_r^s\beta_t^r + \beta_r^s\beta_t^r)\Psi^t \end{aligned}$$

for $s \neq r \neq t$. Symmetry of the second derivatives gives

$$\begin{aligned} &\frac{\partial\alpha_r^s}{\partial R^t} = \frac{\partial\alpha_t^s}{\partial R^r}, \quad s \neq t \neq r, \\ &\frac{\partial\beta_r^s}{\partial R^t} + \beta_r^s\alpha_t^r = \alpha_t^s\beta_r^s + \beta_t^s\beta_r^t. \end{aligned} \tag{6.23}$$

We see that the conditions of the symmetry of the second derivatives are the algebraic consequence of the system (6.22), because the formulas (6.23) are identically satisfied by virtue of the relations (6.6) and (6.7).

If we make the transformation (6.8), then Eq. (6.22) can be written in a more simple form

$$\frac{\partial\Psi^s}{\partial R^r} = \beta_r^s\Psi^r, \quad \text{where } \beta_r^s = \beta_r^s(R). \tag{6.24}$$

Notice that in the formula (6.24) the variable x does not appear explicitly.

Remark: As we know, the solution of Eq. (6.4) can be obtained by integration of the system (6.22). Conversely, we will show that every nondegenerated solution of the system (6.4) can be obtained using this method.

Indeed, Eq. (6.4) implies that for $\mathscr{O}^s(x) := \lambda^s_\mu(R(x))x^\mu$ we have

$$d\mathscr{O}^s = \lambda^s(R) + \sum_{r=1}^{k} \frac{\partial(\lambda^s_\mu x^\mu)}{\partial R^r} \xi^r \lambda^r(R).$$

If $\xi^r \not\equiv 0$, then

$$d\mathscr{O}^s = \sum_{r=1}^{k} \varphi^s_r dR^r, \quad \text{for } s = 1,...,k, \tag{6.25}$$

where ψ^s_r are functions determined on $\mathscr{E}$. Additionally, the functions $R^1,...,R^k$ are functionally independent, so we can add them to the map $(R^1,...,R^p,R^{p+1},...,R^n)$. Then (6.25) means that we have the following conditions in this map:

$$\frac{\partial\mathscr{O}^s}{\partial R^r} = \varphi^s_r \quad \text{for } r = 1,...,p \quad \text{and} \quad \frac{\partial\mathscr{O}^s}{\partial R^{p+1}} = \cdots = \frac{\partial\mathscr{O}^s}{\partial R^n} = 0. \qquad \text{Q.E.D.}$$

so $\mathscr{O}^s = \Psi^s(R^1,...,R^p)$.

Now we will formulate the Bäcklund transformation as follows. Let us consider the system (6.22). Let us assume that we can find[36] linearly independent functions $\lambda^s(R) \in \mathscr{E}^*$ (where $\dim\mathscr{E} \geqslant k$), such that Eq. (6.5) holds. For these functions we can construct the system of Eqs. (6.4). Then using the above-described method we can obtain solutions of the basic system (6.22) from the solutions of Eqs. (6.4). By the Bäcklund transformation we understand here the transformation from Eq. (6.22) up to Eq. (6.4), and vice versa.

In the Paper II we shall use notions introduced here and methodology for the case of nonhomogeneous systems of p.d.e. with the particular emphasis on nonlinear superpositions of simple waves and simple states (propagation of a simple wave or many simple waves on a simple state).

[1]A. Grundland, "Riemann invariants for nonhomogeneous systems of first order partial quasilinear differential equations—algebraic aspects. Examples from gasdynamics," Arch. Mech. Stos. **26** (2), 274–296 (1974).

[2]A. Grundland, "Riemann invariants for nonhomogeneous systems of quasi-linear partial differential equations. Conditions of involution," Bull. Acad. Pol. Sci. Sér. Sci. Tech. **22**(4), 177–185 (1974).

[3]This term is taken from gas dynamics.

[4]B. Riemann, "Uber die Fortpflanzung ebener Luftwallen von endlicher Schwingungsweite," Abhandl. Königt. Gas. Wiss., Göttingen **8** (1869).

[5]A. Jeffrey, *Quasilinear Hyperbolic Systems and Waves* (Pitman, London, 1976).

[6]G. Whitham, *Linear and Nonlinear Waves* (Wiley Interscience, New York, 1974).

[7]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley Interscience, New York, London, 1962), Vol. 2.

[8]B. Rozdestvenski and N. Yanenko, *Quasilinear System of Equations and Their Applications to Gas Dynamics* (Moscow, Nauka, Physico-Mathematical Literature, 1968) (in Russian).

[9]M. Burnat, "The hyperbolic double waves," Bull. Acad. Pol. Sci., Sér. Sci. **17**(10), 97–106 (1969).

[10]M. Burnat, *Riemann Invariants. Fluid Dynamics Transactions* (1969), Vol., 4, pp. 17–22.

[11]M. Burnat, "The method of Riemann invariants for multidimensional nonelliptic systems," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **7** (11–12), 37–42 (1969).

[12]Z. Peradzyński, "Nonlinear plane k-waves and Riemann invariants," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **19** (9), pp. 59–66 (1971).

[13]W. Fiszdon and Z. Peradzyński, *Some geometric properties of a system of first-order non-linear partial differential equations. Trends in applications of pure mathematics and mechanics*, edited by Gaetano Fichera (Pitman, New York, 1976).

[14]Z. Peradzyński, "Nonlinear interaction described by partial differential equations," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **22**, 5 (1974).

[15]Z. Peradzyński, *Geometry and Nonlinear Partial Differential Equations* (Ossolineum, Warsaw, 1979) (in Polish).

[16]Z. Peradzyński, "Asymptotic decay of solutions of hyperbolic system into simple waves," Bull. Acad. Pol. Sér. Sci. Tech. **22**, 513–514 (1974).

[17]W. Zajączkowski, "Riemann invariants interactions for nonelliptic system," Demonstratio Math. **13** (1), 7–21 (1980).

[18]W. Zajączkowski, "Some problem of double waves in magnetohydrodynamics," Arch. Mech. **26** (2), 211–220 (1974).

[19]M. Burnat, Theory of simple waves for nonlinear systems of partial differential equations and applications to gas dynamics. Arch. Mech. Stos. **18**, 4 (1966).

[20]Z. Peradzyński, "On algebraic aspects of the generalized Riemann invariants method," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **18**, 341–346 (1970).

[21]Denoted also by $T_x u \equiv d_x u \equiv du(x)$.

[22]J. Dieudonné, *Foundations of Modern Analysis* (Academic, New York, 1960).

[23]To simplify the formulae, we use the following notation

$$\dot\varphi := \frac{d\varphi(x)}{dx}, \quad R_{,x^\mu} := \frac{\partial R}{\partial x^\mu}, \quad \gamma_{,\gamma'} := \gamma_{,u}:\gamma^l + \gamma_{,\lambda}\,\lambda_{,\gamma}, \lambda_{,\gamma} := \lambda_{,u}:\gamma^l.$$

[24]M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).

[25]N. Ahiezer, *Elements of Elliptic Function Theory* (Nauka, Physico-Mathematical Literature, Moscow, 1970) (in Russian).

[26]A. Actor, "Classical solutions of SU(2) Yang–Mills theories," Rev. Mod. Phys. **51**(3) (1979).

[27]D. Kurdgelaidze, Vestn. Mos. Univ. **8** (1954).

[28]Y. Liouville, J. Math. Pures Appl. **18** (1853).

[29]G. Jorjadze, A. Pogrebkov, and M. Polivanov, Theoret. Math. Phys. **40**, 2 (1979) (in Russian).

[30]R. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972).

[31]A. Gaponov and M. Grehob, *Nonlinear Waves* (Nauka, Physico-Mathematical Literature, Moscow, 1979) (in Russian).

[32]Let $A_s(R) = (A^1_s(R),..., A^n_s(R))$ be the set of the vectors with the property $(A_s(R), \lambda^r(R)) = \delta^r_s$; then from Eqs. (3.10) we can determine the functions α^r_s, β^r_s, i.e., $\alpha^r_s = (\partial\lambda^r/\partial R^s, A_s), \beta^r_s = (\partial\lambda^r/\partial R^s, A_r)$.

[33]After this work was completed we learned that the proof of existence and uniqueness of the solution of the system (6.5) was given also by Z. Peradzyński.[15] However, our proof given in this paper is shorter and simpler.

[34]F. John, *Partial Differential Equations* (Springer-Verlag, New York, 1975).

[35]Analogous conditions for the case of two waves were introduced by Z. Peradzyński.[15]

[36]If the system (6.22) is integrable, i.e., the equations (6.6) and (6.7) are satisfied then if $\mu = 1,...,n$, where $n \geqslant k$, we can obtain a solution of the Eq. (6.5) satisfying the assumed condition at the point $R = R_0$, e.g., $\lambda^s_\mu(R_0) = \delta^s_\mu$. The forms $\lambda^s(R) = (\lambda^s_1(R),...,\lambda^s_n(R))$ satisfy the system (6.5) and are linearly independent in the neighborhood of R_0.

Simple waves in quasilinear hyperbolic systems.[a] II. Riemann invariants for the problem of simple wave interactions

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In this paper a generalization of the Riemann invariant method to the case of a nonhomogeneous system of equations has been formulated. We have discussed in detail the necessary and sufficient conditions for the existence of Riemann invariants. We perform the analysis using the apparatus of differential forms and Cartan theory of systems in involution. The problem has been reduced to examining Pfaff forms. We have considered the connections between the structure of the set of integral elements and the possibility of a construction of special classes of solutions depending on k arbitrary functions of one variable. These solutions can be interpreted physically as the interactions between k simple waves on a simple state. We have proven that, in the case of interaction of many simple waves described by Riemann invariants, a conservation law for the type and quantity of waves holds. It has been also shown that such a solution, resulting from the interaction of many simple waves propagating on the simple state, decay for a large time in an exact way into simple waves (of the same kind as those entering the interaction) on the state. The Cauchy problem for the nonlinear superposition of k-sample waves has been formulated. A couple of theorems useful for this problem have been given in the Sec. III. The functorial properties of the system of equations determining Riemann invariants have been described. The last part of the work contains an analysis of some examples of the solutions of nonhomogeneous magnetohydrodynamic equations from the point of view of the method described above.

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I. INTRODUCTION

Paper II is a continuation of the Paper I[1] of two papers devoted to problems connected with the theory of simple waves and simple states in systems described by partial nonelliptic differential equations[2]:

$$a_j^{s\mu}(u^1,\ldots,u^l)\frac{\partial}{\partial x^\mu}u^j(x)=b^s(u^1,\ldots,u^l), \tag{1.1}$$

$$s=1,\ldots,m,\quad j=1,\ldots,l,\quad \mu=1,\ldots,n,$$

$$x=(x^1,\ldots,x^n)\in\mathscr{E}\subset\mathbb{R}^n,$$

$$u(x)=(u^1(x),\ldots,u^l(x))\in\mathscr{H}\subset\mathbb{R}^l.$$

All notions and notations are taken from Paper I, where they are more formally introduced. Only new concepts are to be introduced here in this paper.

In contrast to the Paper I in which emphasis was put on the notion of simple waves and simple states as the elementary solutions of equations of interest, here in Paper II we are interested in looking after more general classes of solutions and their properties. We consider here the nonlinear superpositions of simple waves described by nonhomogeneous systems of Eqs. (1.1). We are specifically interested in such solutions from the point of view of their degree of freedom contained in the form of arbitrary functions. For example, we consider the case of solutions depending on k-arbitrary functions of one variable.

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II. NONLINEAR SUPERPOSITION OF THE SIMPLE WAVES IN NONHOMOGENEOUS SYSTEMS

The algebraization, which was done in the Sec. II of Paper I, of the system of Eqs. (1.1), allows us to construct more general classes of solutions.[3] We propose now a form of the solution u for which the tangent mapping $du(x)$ is the sum of the homogeneous and a nonhomogeneous simple elements:

$$du(x)=\sum_{t=1}^{k}\xi^t\gamma_t\otimes\lambda^t+\xi^0\gamma_0\otimes\lambda^0,\quad k\leqslant l, \tag{2.1}$$

$$a_j^{s\mu}\gamma_{(t)}^j\lambda^t{}_\mu=0,\quad a_j^{s\mu}\gamma_{(0)}^j\lambda^0=b^s,\quad \xi^0:=1,$$

where $\xi^t\not\equiv 0$ are treated as the functions of x. The simple integral elements ($\gamma_t\otimes\lambda^t$ which are homogeneous and $\gamma_0\otimes\lambda^0$ which is nonhomogeneous) represent the simple waves and a simple state, respectively. Such a solution u of the system (1.1) will be called later on a superposition of the simple waves on a simple state.

It was proved[4-6] that the solutions of the postulated form (2.1) exist and can be written in Riemann invariants. It means that in the latter case (Riemann invariants) we make the additional assumption that the commutator for all the vector fields γ_α and γ_β are the linear combinations of these fields, i.e.,

$$\bigwedge_{\alpha,\beta\in\{0,1,\ldots,k\}}[\gamma_\alpha,\gamma_\beta]\in\operatorname{span}\{\gamma_\alpha,\gamma_\beta\}. \tag{2.2}$$

When these conditions are satisfied, then the vectors $\gamma_0,\gamma_1,\ldots,\gamma_k$ constitute a holonomic system, and there exists a

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parametrization of the surface tangent to the fields $\gamma_0,...,\gamma_k$:

$$u = f(R), \quad \text{where } R := (R^0,...,R^k) \tag{2.3}$$

such that[7]

$$\bigwedge_{\alpha\in\{0,1,...,k\}} \frac{\partial f(R)}{\partial R^\alpha} \sim \gamma_\alpha(R) \in T_{i(R)}H. \tag{2.4}$$

Consequently,

$$du(x) = \sum_{\alpha=0}^{k} \frac{\partial f}{\partial R^\alpha} dR^\alpha, \quad dR^\alpha = \frac{\partial R^\alpha}{\partial x^\mu} dx^\mu, \tag{2.5}$$

which together with the assumption that $\gamma_0,...,\gamma_k$ are linearly independent, leads to the system of Pfaff forms:

$$dR^\alpha(x) = \xi^\alpha \lambda^\alpha(R), \quad \text{where } \xi^\alpha \not\equiv 0, \alpha\in\{0,1,...,k\}. \tag{2.6}$$

The fields of covectors λ^α become functions of the parameter $R := (R^1,...,R^k)$ i.e., $\lambda^\alpha = \lambda^\alpha_\mu(R)dx^\mu \in \mathscr{E}^*$). Involutivity conditions for the system (2.6) for the case of many independent variables were already investigated in Refs. 4–6. These conditions put some restrictions on the covectors λ^α in the system (2.6). Namely, the system (2.6) has solutions (is completely integrable) if the following conditions are satisfied:

$$\frac{\partial \lambda^0(R)}{\partial R^\sigma} = \alpha_\sigma \lambda^\sigma(R), \quad \sigma\in\{0,...,k\}, \quad r\in\{1,...,k\}, \tag{2.7a}$$

$$\frac{\partial \lambda^r(R)}{\partial R^\sigma} = \beta^r_\sigma \lambda^r(R) + \zeta^r_\sigma \lambda^\sigma(R), \quad \sigma\neq r, \tag{2.7b}$$

$$\lambda^{\alpha_1} \wedge \lambda^{\alpha_2} \wedge \lambda^{\alpha_3} \neq 0 \quad \text{for } \alpha_1 < \alpha_2 < \alpha_3,$$

where $\alpha_\sigma, \beta^r_\sigma, \zeta^r_\sigma$ are given functions[8] of R and we have not performed summation with respect to the index σ. These relations are necessary and sufficient conditions for the existence of the solutions of the system (2.6). They assure us[5,6] that the set of solutions of the system (2.6) depends on k arbitrary functions of one variable according to Cartan theory of systems in involution.

Let us consider separately two extreme cases: (1) when the coefficients α_s do not vanish anywhere and (2) when α_s are identically equal to zero.

A. The case $\alpha_r \neq 0$

Let us assume that all the functions α_r are different from zero; then it follows from Eqs. (2.7a) that

$$\lambda^r = \frac{1}{\alpha_r} \frac{\partial \lambda^0}{\partial R^r}, \quad r\in\{1,...,k\}. \tag{2.8}$$

Let us notice that Eqs. (2.7b) for $\sigma = 0$ are the consequence of Eqs. (2.7a)

$$\begin{aligned}\frac{\partial \lambda^r}{\partial R^0} &= \frac{\partial}{\partial R^0}\left(\frac{1}{\alpha_r}\frac{\partial \lambda^0}{\partial R^r}\right) \\ &= -\frac{1}{\alpha_r}\frac{\partial \alpha_r}{\partial R^0}\lambda^r + \frac{1}{\alpha_r}\frac{\partial}{\partial R^r}(\alpha_0 \lambda^0) \\ &= \frac{1}{\alpha_r}\frac{\partial \alpha_0}{\partial R^r}\lambda^0 + \left(\alpha_0 - \frac{1}{\alpha_r}\frac{\partial \alpha_r}{\partial R^0}\right)\lambda^r.\end{aligned}$$

Function λ^0 must satisfy Eq. (2.7a) for $\sigma = 0$, i.e.,

$$\frac{\partial \lambda^0}{\partial R^0} = \alpha_0 \lambda^0.$$

Let the function φ be the solution of the equation $\partial\varphi/\partial R^0 = \alpha_0$; then

$$\lambda^0(R) = \lambda(\tilde{R})\exp[\varphi(R)], \tag{2.9a}$$

$$\lambda^r(R) \sim \frac{\partial \lambda^0}{\partial R^r} \sim \left(\frac{\partial \varphi(R)}{\partial R^r}\lambda(\tilde{R}) + \frac{\partial \lambda(\tilde{R})}{\partial R^r}\right), \tag{2.9b}$$

where $\tilde{R} := (R^1,...,R^k)$. Moreover, the conditions (2.7a) give

$$\begin{aligned}\frac{\partial^2 \lambda^0}{\partial R^r \partial R^s} &= \frac{\partial}{\partial R^r}(\alpha_s \lambda^s) = \left(\frac{\partial \alpha_s}{\partial R^r} + \alpha_s \beta^s_r\right)\lambda^s + \alpha_s \zeta^s_r \lambda^r \\ &= C_{rs}\frac{\partial \lambda^0}{\partial R^s} + C_{rs}\frac{\partial \lambda^0}{\partial R^r}, \quad r\neq s\in\{1,...,k\},\end{aligned} \tag{2.10}$$

where $C_{rs} := (1/2\alpha_s)(\partial\alpha_s/\partial R^r + \alpha_s\beta^s_r + \alpha_r\zeta^r_s)$. Inserting (2.8) and (2.9a) into Eqs. (2.10), we obtain linear hyperbolic system of equations of second order for the unknown function λ, i.e.,

$$\frac{\partial^2 \lambda}{\partial R^s \partial R^r} = P_{rs}\frac{\partial \lambda}{\partial R^s} + P_{sr}\frac{\partial \lambda}{\partial R^r} + Q_{rs}\lambda, \quad r\neq s, \tag{2.11a}$$

where the quantities

$$\begin{aligned}P_{rs} &:= C_{rs} - \partial\varphi/\partial R^r, \quad Q_{rs} := C_{rs}\partial\varphi/\partial R^s \\ &\quad + C_{sr}\partial\varphi/\partial R^r - \partial^2\varphi/\partial R^r \partial R^s\end{aligned}$$

are treated as the given functions of R in the domain $\mathscr{D} \subset \mathscr{H}$ having continuous second derivatives.

From Eqs. (2.11a) we have

$$\begin{aligned}\frac{\partial^3 \lambda}{\partial R^r \partial R^s \partial R^t} &= \left(\frac{\partial P_{rs}}{\partial R^t} + P_{rs}P_{st}\right)\frac{\partial \lambda}{\partial R^s} \\ &\quad + (P_{rs}P_{ts} + P_{sr}P_{tr} + Q_{rs})\frac{\partial \lambda}{\partial R^t} \\ &\quad + \left(\frac{\partial P_{sr}}{\partial R^t} + P_{sr}P_{rt}\right)\frac{\partial \lambda}{\partial R^r} \\ &\quad + \left(\frac{\partial Q_{rs}}{\partial R^t} P_{rs}Q_{st} + P_{sr}Q_{rt}\right)\lambda, \quad r\neq s\neq t.\end{aligned}$$

Hence the compatibility conditions are of the form

$$\begin{aligned}&\frac{\partial P_{rt}}{\partial R^s} + P_{rt}P_{st} - P_{rs}(P_{st} + P_{rt}) - Q_{rs} = 0, \\ &\frac{\partial P_{sr}}{\partial R^t}\frac{\partial P_{tr}}{\partial R^s} = 0, \quad r\neq s\neq t, \\ &\frac{\partial Q_{rs}}{\partial R^t} - \frac{\partial Q_{rt}}{\partial R^s} + Q_{st}(P_{rs} - P_{rt}) + Q_{rt}P_{sr} - Q_{rs}P_{tr} = 0.\end{aligned} \tag{2.11b}$$

Equations (2.11) separate into n independent scalar equations for the components of the vector λ^s in the form

$$\frac{\partial^2 u}{\partial R^s \partial R^r} = P_{rs}\frac{\partial u}{\partial R^s} + P_{sr}\frac{\partial u}{\partial R^r} Q_{sr}u, \quad s\neq r. \tag{2.12}$$

In Eqs. (2.12) independent variables R^i ($i\notin\{r,s\}$) can be treated as parameters. Thus we obtain one hyperbolic equation with two independent variables. The Cauchy problem for Eq. (2.12) can be solved by means of the Riemann function. One can prove[9] that the solution of Eq. (2.12) exists and is unique in the rectangle bounded by characteristics containing the noncharacteristic initial curve.

If the covector $\lambda^0(R)$ is of the form (2.9a) and the covector λ^r is determined by formula (2.9b), then compatibility conditions (2.7) reduce to Eqs. (2.11). So we will investigate the system of Eqs. (2.6) in the form

$$dR^0 = \lambda(\tilde{R})\exp[\varphi(R)], \tag{2.13a}$$

$$dR^s = \xi^s\left(\frac{\partial\varphi(R)}{\partial R^s}\lambda(\tilde{R}) + \frac{\partial\lambda(\tilde{R})}{\partial R^s}\right), \tag{2.13b}$$

(where $\lambda, \partial\lambda/\partial R^1, \ldots, \partial\lambda/\partial R^k \in \mathscr{E}^*$ are linearly independent), assuming that the compatibility conditions (2.11) are satisfied.

We prove now that, assuming (2.11), we can reduce the system of Pfaff forms (2.13) to the overdetermined system of partial differential equations of the second order for one unknown function depending on $k+1$ variables $R^0, \ldots, R^k$. We present the unknown solutions of the system (2.13) in the implicit form[10]

$$\lambda(\tilde{R})\cdot x = \Psi^0(R), \quad \frac{\partial\lambda(\tilde{R})}{\partial R^s}\cdot x = \Psi^s(R). \tag{2.14}$$

Differentiating Eqs. (2.14), we obtain

$$\sum_{s=1}^{k}\frac{\partial\lambda}{\partial R^s}\cdot x\,dR^s + \lambda = \frac{\partial\Psi^0}{\partial R^0}dR^0 + \sum_{s=1}^{k}\frac{\partial\Psi^0}{\partial R^s}dR^s,$$

$$\sum_{s\neq r=1}^{k}\frac{\partial^2\lambda}{\partial R^r\partial R^s}\cdot x\,dR^s + \frac{\partial^2\lambda}{\partial R^r\partial R^r}\cdot x\,dR^r + \frac{\partial\lambda}{\partial R^r}$$

$$= \frac{\partial\Psi^r}{\partial R^0}dR^0 + \sum_{s\neq r=1}^{k}\frac{\partial\Psi^r}{\partial R^s}dR^s + \frac{\partial\Psi^r}{\partial R^r}dR^r. \tag{2.15}$$

If the system (2.13) is satisfied, then for $\xi^s \not\equiv 0$ we have

$$\sum_{s=1}^{k}\frac{\partial\lambda}{\partial R^s}\cdot x\,dR^s + \exp(-\varphi)\,dR^0$$

$$= \frac{\partial\Psi^0}{\partial R^0}dR^0 + \sum_{s=1}^{k}\frac{\partial\Psi^0}{\partial R^s}dR^s,$$

$$\sum_{s\neq r=1}^{k}\frac{\partial^2\lambda}{\partial R^r\partial R^s}\cdot x\,dR^s + \frac{\partial^2\lambda}{\partial R^r\partial R^r}\cdot x\,dR^r$$

$$+ \frac{1}{\xi^r}dR^r - \frac{\partial\varphi}{\partial R^r}\exp(-\varphi)\,dR^0$$

$$= \frac{\partial\Psi^r}{\partial R^0}dR^0 + \sum_{s\neq r=1}^{k}\frac{\partial\Psi^r}{\partial R^s}dR^s + \frac{\partial\Psi^r}{\partial R^r}dR^r.$$

Assuming that Eqs. (2.13) are nondegenerated (i.e., $dR^0, dR^1, \ldots, dR^k$ are linearly independent; thus $\xi^s \not\equiv 0$), we have

$$\frac{\partial\Psi^0}{\partial R^0} = \exp(-\varphi), \quad \frac{\partial\Psi^r}{\partial R^0} + \frac{\partial\varphi}{\partial R^r}\exp(-\varphi) = 0,$$

$$\frac{\partial\Psi^0}{\partial R^s} = \frac{\partial\lambda}{\partial R^s}\cdot x, \quad \frac{\partial\Psi^r}{\partial R^s} = \frac{\partial^2\lambda}{\partial R^r\partial R^s}\cdot x, \quad r\neq s,$$

$$\frac{\partial\Psi^r}{\partial R^r} - \frac{\partial^2\lambda}{\partial R^r\partial R^r}\cdot x = \frac{1}{\xi^r}.$$

Taking into account Eqs. (2.14) and compatibility conditions (2.11), we get

$$\frac{\partial\Psi^0}{\partial R^0} + = \exp(-\varphi), \tag{2.16a}$$

$$\frac{\partial\Psi^0}{\partial R^s} + = \Psi^s, \quad s,r\in\{1,\ldots,k\}, \tag{2.16b}$$

$$\frac{\partial\Psi^s}{\partial R^0} + \frac{\partial\varphi}{\partial R^s}\exp(-\varphi) = 0, \tag{2.16c}$$

$$\frac{\partial\Psi^r}{\partial R^s} = P_{rs}\Psi^s + P_{sr}\Psi^r + Q_{rs}\Psi^0, \quad r\neq s, \tag{2.16d}$$

$$\frac{\partial\Psi^r}{\partial R^r} - \frac{\partial^2(\lambda\cdot x)}{\partial R^r} = \frac{1}{\xi^r}. \tag{2.16e}$$

Note that Eqs. (2.16c) are the closure conditions for the 1-forms $\Omega^s := \exp(-\varphi)\,dR^0 + \Psi^s\,dR^s$, which are the compatibility conditions of the system (2.16a) and (2.16b) for the function Ψ^0. The general solution of Eqs. (2.16c) has the form

$$\Psi^s(R) = \frac{\partial\Phi(\tilde{R})}{\partial R^s} - \int_0^{R^0}\frac{\partial\varphi(r,\tilde{R})}{\partial R^s}\exp[-\varphi(r,\tilde{R})]\,dr,$$

[where the constant of integration with respect to R^0, depending on $\tilde{R}$, is denoted for convenience by $\partial\Phi(\tilde{R})/\partial R^s$]. Having the functions $\Psi^s(R)$, we can solve the system of Eqs. (2.16a) and (2.16b). Integrating the closed forms Ω^s, we have

$$\Psi^0(R) = \int_0^{R^s}\frac{\partial\Phi(\tilde{R})}{\partial R^s}dR^s + \int_0^{R^0}\exp[-\varphi(r,\tilde{R})]\,dr.$$

Thus the general solution of the system (2.16a)–(2.16c) has the form

$$\Psi^0(R) = \Phi(\tilde{R}) + \int_0^{R^0}\exp[-\varphi(r,\tilde{R})]\,dr,$$

$$\Psi^s(R) = \frac{\partial\Phi(\tilde{R})}{\partial R^s} - \int_0^{R^0}\frac{\partial\varphi(r,\tilde{R})}{\partial R^s}\exp[-\varphi(r,\tilde{R})]\,dr. \tag{2.17}$$

Inserting (2.16b) into Eqs. (2.16d), we get

$$\frac{\partial^2\Psi^0}{\partial R^r\partial R^s} = P_{rs}\frac{\partial\Psi^0}{\partial R^s} + P_{sr}\frac{\partial\Psi^0}{\partial R^r} + Q_{rs}\Psi^0, \quad s\neq r. \tag{2.18}$$

So conditions (2.18) constitute an overdetermined system of $k^2 - k$ partial differential equations of the second order for one unknown function Ψ^0 depending on $k+1$ variables $R^0, \ldots, R^k$ (while the variable R^0 is treated as a parameter).

Then Eqs. (2.16e) give

$$\left(\frac{\partial\Psi^r}{\partial R^r} - \frac{\partial^2(\lambda^r\cdot x)}{\partial(R^r)^2}\right)dR^r = \lambda^r$$

$$= \left(\frac{\partial\varphi(R)}{\partial R^r}\lambda(\tilde{R}) + \frac{\partial\lambda(\tilde{R})}{\partial R^r}\right),$$

where $\partial\Psi^r/\partial R^r - \partial^2(\lambda^r\cdot x)/\partial(R^r)^2 \neq 0$ are the resolvability condition. So we have (2.13b).

Remark 1: For $k=1$ the compatability conditions (2.18) do not exist. So the solution of the system (2.13) is determined by formulae (2.14) and (2.17). Let us notice that for an arbitrary $k\geqslant 1$ if $\Psi^0 = \sum_{\mu=1}^{n}\sigma^\mu\lambda_\mu$, $\sigma^\mu\in\mathbb{R}^1$, then by virtue (2.11) the Eqs. (2.18) are automatically satisfied.

Theorem 1: If the conditions (2.11) are satisfied, then the general integral of the system (2.13) has the implicit form

$$\lambda(\tilde{R})\cdot x = \Phi(\tilde{R}) + \int_0^{R^0}\exp[-\varphi(r,\tilde{R})]\,dr, \tag{2.19a}$$

$$\frac{\partial\lambda(\tilde{R})\cdot x}{\partial R^s} = \frac{\partial\Phi(R)}{\partial R^s} - \int_0^{R^0}\frac{\partial\varphi(r,\tilde{R})}{\partial R^s}\exp[-\varphi(r,\tilde{R})]\,dr, \tag{2.19b}$$

where the function $\Phi(\tilde{R})$ satisfies

$$\frac{\partial^2\Phi}{\partial R^r\partial R^s} - P_{rs}\frac{\partial\Phi}{\partial R^s} - P_{sr}\frac{\partial\Phi}{\partial R^r} - Q_{rs}\Phi$$
$$= \int_0^{R^0}\left(\frac{\partial^2\varphi}{\partial R^r\partial R^s} - \frac{\partial\varphi}{\partial R^r}\frac{\partial\varphi}{\partial R^s}\right)\exp(-\varphi)\,dr$$
$$- P_{rs}\int_0^{R^0}\frac{\partial\varphi}{\partial R^s}\exp(-\varphi)\,dr - P_{sr}$$
$$\times\int_0^{R^0}\frac{\partial\varphi}{\partial R^r}\exp(-\varphi)\,dr$$
$$+ Q_{rs}\int_0^{R^0}\exp(-\varphi)\,dr \quad \text{for } s\neq r\in\{1,\dots,k\}. \tag{2.20}$$

Proof: We show that if the conditions (2.11) are satisfied, then the implicit form of Eqs. (2.19) determines the solution of the system (2.13). Indeed, differentiating Eq. (2.19a), we get

$$\sum_{p=1}^{k}\left[\frac{\partial\lambda\cdot x}{\partial R^p} - \frac{\partial\Phi}{\partial R^p} + \int_0^{R^0}\frac{\partial\varphi}{\partial R^p}\exp(-\varphi)\,dr\right]$$
$$\times dR^p + \lambda = \exp(-\varphi)\,dR^0. \tag{2.21}$$

Hence by virtue of (2.19b), we have $\lambda = \exp(-\varphi)\,dR^0$, that is, (2.13a). Next, differentiating Eqs. (2.19b), we obtain

$$\sum_{p=1}^{k}\left[\frac{\partial^2\lambda\cdot x}{\partial R^s\partial R^p} - \frac{\partial^2\Phi}{\partial R^s\partial R^p}\right.$$
$$\left. + \int_0^{R^0}\left(\frac{\partial^2\varphi}{\partial R^s\partial R^p} - \frac{\partial\varphi}{\partial R^s}\frac{\partial\varphi}{\partial R^p}\right)\exp(-\varphi)\,dr\right]dR^p$$
$$+ \frac{\partial\lambda}{\partial R^s} = \frac{\partial\varphi}{\partial R^s}\exp(-\varphi)\,dR^0. \tag{2.22}$$

Let us notice that by virtue of (2.11) and (2.20) the matrix

$$A_{sp} := \frac{\partial^2\lambda\cdot x}{\partial R^s\partial R^p} - \frac{\partial^2\Phi}{\partial R^s\partial R^p}$$
$$+ \int_0^{R^0}\left(\frac{\partial^2\varphi}{\partial R^s\partial R^p} - \frac{\partial\varphi}{\partial R^s}\frac{\partial\varphi}{\partial R^p}\right)\exp(-\varphi)\,dr$$

is diagonal. Hence, utilizing Eq. (2.13a) just derived, we have

$$dR^s = -\frac{1}{A_{ss}}\left[\frac{\partial\lambda}{\partial R^s} + \frac{\partial\varphi}{\partial R^s}\exp(-\varphi)\,dR^0\right]$$
$$= \frac{-1}{A^{ss}}\left(\frac{\partial\varphi}{\partial R^s}\lambda + \frac{\partial\lambda}{\partial R^s}\right),$$

that is, (2.13b).

Conversely, we show that every nondegenerated solution $R(x)$ of the system (2.13) can be obtained by resolution with respect to the variables $R^0,\dots,R^k$ of the system (2.19). Indeed, if (2.13) is satisfied, then

$$d[\lambda(\tilde R)\cdot x] = \sum_{a=1}^{k}\frac{\partial\lambda(\tilde R)\cdot x}{\partial R^a}dR^a + \lambda(\tilde R)$$
$$= \exp(-\varphi)\,dR^0 + \sum_{a=1}^{k}\frac{\partial\lambda(\tilde R)\cdot x}{\partial R^a},$$

$$d\left[\frac{\partial\lambda(\tilde R)\cdot x}{\partial R^s}\right] = \sum_{a=1}^{k}\frac{\partial^2\lambda(\tilde R)\cdot x}{\partial R^s\partial R^a}dR^a$$
$$+ \frac{\partial\lambda(\tilde R)}{\partial R^s} = -\frac{\partial\varphi}{\partial R^s}\exp(-\varphi)\,dR^0$$
$$+ \sum_{s\neq a=1}^{k}\frac{\partial^2\lambda(\tilde R)\cdot x}{\partial R^s\partial R^a}dR^a$$
$$+ \left(\frac{\partial^2\lambda(\tilde R)\cdot x}{\partial R^s\partial R^s} + \frac{1}{\xi^s}\right)dR^s.$$

So the function $\lambda(\tilde R)\cdot x$ and $[\partial\lambda(\tilde R)\cdot x]/\partial R^s$ can be expressed as functions of $R^0(x),\dots,R^k(x)$. Q.E.D.

Now we will formulate the Bäcklund transformation as follows. Let us consider the system (2.20). Let us assume that we can find[11] function $\lambda\in\mathscr{E}^*$ such that $\lambda\wedge\partial\lambda/\partial R^1\wedge\cdots\wedge\partial\lambda/\partial R^k\neq 0$ (where $\dim\mathscr{E}\geqslant k$) and that Eqs. (2.11a) hold. For this function $\lambda(\tilde R)$ we can construct the system of Eqs. (2.13). Then, using the above-described method, we can obtain solutions of the basic system (2.20) from the solutions of Eqs. (2.13). By the Bäcklund transformation we mean the transformation from Eqs. (2.20) up to Eqs. (2.13) and vice versa.

B. The case when all $\alpha_r = 0$

Let us notice that in this case it follows from Eqs. (2.7a) that

$$\lambda^0(R) = C\exp[\varphi(R^0)], \quad \text{where } C\in\mathscr{E}^*,$$

where $\varphi(R^0)$ is a differentiable function of one variable. When $\sigma = 0$ by virtue of Eq. (2.7b) for arbitrary $\rho^r = \rho^r(R)$ and $\kappa^r = \kappa^r(R)$, we have

$$\frac{\partial}{\partial R^0}[\lambda^r\exp(-\rho^r) - \kappa^r C]$$
$$= \left[\beta^r_0\exp(-\rho^r+\varphi) - \frac{\partial\kappa^r}{\partial R^0}\right]C$$
$$+ \left(\zeta^s_0 - \frac{\partial\rho^r}{\partial R^0}\right)\exp(-\rho^r)\lambda^r.$$

So if for quantities ρ^s and κ^s, we take the solutions of the system

$$\frac{\partial\rho^r}{\partial R^0} = \zeta^r_0, \quad \frac{\partial\kappa^r}{\partial R^0} = \beta^r_0\exp(\varphi-\rho)$$

(which always locally exist), then

$$\lambda^r(R)\exp[-\rho^r(R)] = \kappa^r(R)C + A^r(\tilde R),$$
$$\tilde R := (R^1,\dots,R^k),$$

where $A:\mathscr{H}\to\mathscr{E}^*$ are differentiable functions of the class C^1. So we have

$$\lambda^0(R) = C\exp[\varphi(R^0)],$$
$$\lambda^r(R)\sim\{\kappa^r(R)C + A^r(\tilde R)\}, \quad r\in\{1,\dots,k\}. \tag{2.23}$$

Inserting (2.23) into Eqs. (2.7b) for $\sigma>0$, together with the assumption that $C, A^1,\dots,A^k$ are linearly independent, we get

$$\frac{\partial A^r}{\partial R^t} = (\beta^r_t\kappa^t - \frac{\partial\kappa^r}{\partial R^t}$$
$$+ \zeta^r_t)C + \beta^r_t A^t + \zeta^r_t A^r, \quad r\neq t\in\{1,\dots,k\}. \tag{2.24}$$

Thus in this case the system (2.6) has the form

$$d\chi(R^0) = C, \quad \text{where } \chi(R^0) := \int_0^{R^0}\exp[-\varphi(r)]\,dr, \tag{2.25a}$$

$$dR^r = \xi^r[\kappa^r(R)C + A^r(\tilde R)], \quad r\in\{1,\dots,k\}, \tag{2.25b}$$

for which the compatibility conditions are given by (2.24). From Eq. (2.25a) it follows immediately that $\chi(R^0) = C_\mu x^\mu + C_0$, $C_0\in\mathbb{R}^1$. Let us notice that if the variables R^r satisfy Eqs. (2.25b) in which $R^0 = \chi^{-1}(C_\mu x^\mu + C_0)$,

then

$$d\left[\int_0^{R^0}\kappa^r(r,\tilde R)e^{-\varphi(\chi(r))}\,dr^0+A^r_\mu(\tilde R)x^\mu\right]$$
$$=\kappa^r(R^0,\tilde R)C+A^r(\tilde R)x$$
$$+\left[\int_0^{R^0}\frac{\partial\kappa^r(r,\tilde R)}{\partial R^s}e^{-\varphi(\chi(r))}\,dr^0+\frac{\partial A^r_\mu x^\mu}{\partial R^r}\right]dR^r$$
$$+\sum_{r\neq t=1}^{k}\left[\int_0^{R^0}\frac{\partial\kappa^r(r,\tilde R)}{\partial R^t}e^{-\varphi(\chi(r))}\,dr^0+\frac{\partial A^r_\mu x^\mu}{\partial R^t}\right]dR^t$$
$$=\left[\frac{1}{\xi^r}+\int_0^{R^0}\frac{\partial\kappa^r(r,\tilde R)}{\partial R^r}e^{-\varphi(\chi(r))}\,dr^0+\frac{\partial A^r_\mu x^\mu}{\partial R^r}\right]dR^r$$
$$+\sum_{r\neq t=1}^{k}\left[\int_0^{R^0}\frac{\partial\kappa^r(r,\tilde R)}{\partial R^t}e^{-\varphi(\chi(r))}\,dr^0+\frac{\partial A^r_\mu x^\mu}{\partial R^t}\right]dR^t.$$

If $\xi^r\not\equiv 0$, then the functions $[\int_0^{R^0}\kappa^r(r,\tilde R)e^{-\varphi(\chi(r))}\,dr^0+A^r_\mu(\tilde R)x^\mu]$ can be expressed as the functions of $R^1(x),\dots,R^k(x)$. So we have

$$\int_0^{R^0}\kappa^s(r,\tilde R)e^{-\varphi(\chi(r))}\,dr^0+A^s_\mu(\tilde R)x^\mu=\Psi^s(\tilde R),\quad s\in\{1,\dots,k\},$$

where Ψ^s are differentiable functions of k variables. We have the following theorem.

Theorem 2: If the conditions (2.24) are satisfied, then the general integral of the system (2.25) has the implicit form

$$C_\mu x^\mu+C_0=\int_0^{R^0}\exp[-\varphi(r)]\,dr, \tag{2.26a}$$

$$\int_0^{R^0}\kappa^r(r,\tilde R)e^{-\varphi(\chi(r))}\,dr^0+A^r_\mu(\tilde R)x^\mu=\Psi^r(\tilde R),$$

$$s\in\{1,\dots,k\}, \tag{2.26b}$$

where $\Psi^s(\cdot)$ are arbitrary differentiable functions of $\tilde R$.

Remark 2: Similarly, as in the previous case for $k=1$, the compatibility conditions (2.24) do not exist. So the solution of the system (2.25) is determined by formulae (2.26).

Remark 3: In the particular case when in Eqs. (2.25b) the functions $A^1(R^1),\dots,A^k(R^k)\in\mathscr{E}^*$ are arbitrary functions of their arguments, then the compatibility conditions (2.24) are automatically satisfied, and κ^s are arbitrary functions of R^0 and R^s. So we can study the system (2.25) in the form

$$dR^0=C\exp[\varphi(R^0)],$$
$$dR^s=\xi^s\{\kappa^s(R^0,R^s)C+A^s(R^s)\},\quad s\in\{1,\dots,k\}, \tag{2.27}$$

where $C,A^1(R^1),\dots,A^k(R^k)\in\mathscr{E}^*$ are linearly independent. The general integral of the system (2.27) has the form

$$C_\mu x^\mu+C_0=\int_0^{R^0}\exp[-\varphi(r^0)]\,dr^0,$$

$$\int_0^{R^0}\kappa^s(r^0,R^s)\exp[-\varphi(\chi(r))]\,dr^0+A^s_\mu(R^s)x^\mu=\Psi^s(R^s),$$

where $\Psi^1(\cdot),\dots,\Psi^k(\cdot)$ are arbitrary functions of their arguments. Other cases, when only some α^r do vanish should be considered analogously.

III. FORMULATION OF THE CAUCHY PROBLEM

Let us study now an example of the formulation of the Cauchy problem for the Pfaff system of the form

$$dR^0=\xi^0\lambda^0(R),$$

where

$$\lambda^0(R)=\lambda(\tilde R)\exp[\varphi(R)],\quad \xi^\alpha(x)\not\equiv 0,\quad \alpha\in\{0,1,\dots,k\}, \tag{3.1}$$

$$dR^s=\xi^s\lambda^s(R),\quad \lambda^s(R)=\lambda(\tilde R)\varphi_{,R^s}(R)+\lambda_{,R^s}(\tilde R),$$
$$s\in\{1,\dots,k\}.$$

According to Sec. II, the system (3.1) has solutions if the conditions (2.11) are satisfied. We are looking for the solution in the form

$$\lambda(\tilde R)\cdot x=G^0(R),\quad \lambda(\tilde R)\cdot x=G^s(R). \tag{3.2}$$

Then we have

$$\sum_{a=1}^{k}\frac{\partial\lambda}{\partial R^a}\cdot x\,dR^a+\lambda=\frac{\partial G^0}{\partial R^0}dR^0+\sum_{a=1}^{k}\frac{\partial G^0}{\partial R^a}dR^a,$$

$$\sum_{a\neq s=1}^{k}\frac{\partial\lambda}{\partial R^s\partial R^a}x\,dR^a+\frac{\partial\lambda}{\partial R^s\partial R^s}x\,dR^s+\frac{\partial\lambda}{\partial R^s}$$
$$=\frac{\partial G^s}{\partial R^0}dR^0+\sum_{a\neq s=1}^{k}\frac{\partial G^s}{\partial R^a}dR^a+\frac{\partial G^s}{\partial R^s}dR^s.$$

If Eqs. (3.1) are satisfied, then for $\xi^\alpha\not\equiv 0$ we have

$$\frac{1}{\xi^0}\exp(-\varphi)\,dR^0+\sum_{a=1}^{k}\frac{\partial\lambda}{\partial R^a}x\,dR^a$$
$$=\frac{\partial G^0}{\partial R^0}dR^0+\sum_{a=1}^{k}\frac{\partial G^0}{\partial R^a}dR^a,$$

$$\sum_{a\neq s=1}^{k}\frac{\partial\lambda}{\partial R^a\partial R^s}x\,dR^a$$
$$+\frac{\partial\lambda}{\partial R^s\partial R^s}x\,dR^s+\frac{1}{\xi^s}dR^s$$
$$-\frac{\partial\varphi}{\partial R^s}\frac{1}{\xi^0}\exp(-\varphi)\,dR^0$$
$$=\sum_{a\neq s=1}^{k}\frac{\partial G^s}{\partial R^a}dR^a+\frac{\partial G^s}{\partial R^s}dR^s+\frac{\partial G^s}{\partial R^0}dR^0.$$

Assuming that the solutions of Eqs. (3.1) are not degenerate (i.e., $dR^0\wedge dR^1\wedge\cdots\wedge dR^k\neq 0$), we get

$$\frac{\partial G^0}{\partial R^0}=\frac{1}{\xi^0}\exp(-\varphi),\quad \frac{\partial G^s}{\partial R^0}+\frac{\partial\varphi}{\partial R^s}\exp(-\varphi)=0,$$
$$\frac{\partial G^0}{\partial R^a}=\frac{\partial\lambda}{\partial R^a}x,\quad \frac{\partial G^s}{\partial R^s}-\frac{\partial\lambda}{\partial R^s\partial R^s}x=\frac{1}{\xi^s},$$
$$\frac{\partial G^s}{\partial R^a}=\frac{\partial\lambda}{\partial R^a\partial R^s}x,\quad s\neq a.$$

So, utilizing Eqs. (3.2) and compatibility conditions (2.11), we obtain

$$\frac{\partial G^0}{\partial R^0}=\frac{1}{\xi^0}\exp(-\varphi), \tag{3.3a}$$

$$\frac{\partial G^0}{\partial R^a}=G^a, \tag{3.3b}$$

$$\frac{\partial G^s}{\partial R^0}+\frac{\partial\varphi}{\partial R^s}\frac{1}{\xi^0}\exp(-\varphi)=0, \tag{3.3c}$$

$$\frac{\partial G^s}{\partial R^a}=P_{sa}G^a+P_{as}G^s+Q_{sa}G^0, \tag{3.3d}$$

$$\frac{\partial G^s}{\partial R^s}-\frac{\partial\lambda}{\partial R^s\partial R^s}x+=\frac{1}{\xi^s}. \tag{3.3e}$$

Inserting (3.3b) into Eq. (3.3c), we get

$$\frac{\partial G^0}{\partial R^0 \partial R^s} + \frac{\partial \varphi}{\partial R^s}\frac{\partial G^0}{\partial R^0} = 0.$$

So we have $((\partial G^0/\partial R^0)\exp\varphi)_{,R^s} = 0$. Then

$$\frac{\partial G^0}{\partial R^0}(R) = a(R^0)\exp[(-\varphi(R)],$$

where a is an arbitrary function of class C^1 of the variable R^0. Finally we have

$$G^0(R) = \Phi(\tilde{R}) + \int_0^{R^0} a(r)\exp[-\varphi(r,\tilde{R})]\,dr,$$

$$G^s(R) = \frac{\partial \Phi(\tilde{R})}{\partial R^s} - \int_0^{R^0} a(r)\frac{\partial \varphi(r,\tilde{R})}{\partial R^s}\exp[-\varphi(r,\tilde{R})]\,dr, \tag{3.4}$$

where the function $\Phi(\tilde{R})$ satisfy the conditions (2.20).

Let on some regular k-dimensional surface

$$\Gamma = \{x = \zeta(\tilde{\rho})\}, \quad \rho := (\rho^1,\ldots,\rho^k)$$

in the space $\mathscr{E}$ be given a value $R|_\Gamma$, i.e., the initial data have the form

$$R^0(\zeta(\tilde{\rho})) = \rho^0(\tilde{\rho}), \quad R^s(\zeta(\tilde{\rho})) = \rho^s, \tag{3.5}$$

where ρ^0 is treated as a given function of $\tilde{\rho}$. Inserting this into Eqs. (3.2) and (3.4), we get

$$\lambda(\tilde{\rho})\cdot\zeta(\tilde{\rho}) = \Phi(\tilde{\rho}) + \int_0^{\rho^0(\tilde{\rho})} a(r)\exp[-\varphi(r,\tilde{\rho})]\,dr, \tag{3.6a}$$

$$\lambda_{,\rho^s}(\tilde{\rho})\cdot\zeta(\tilde{\rho}) = \Phi_{,\rho^s}(\tilde{\rho}) - \int_0^{\rho^0(\tilde{\rho})} a(r)\varphi_{,\rho^s}(r,\tilde{\rho})\exp[-\varphi(r,\tilde{\rho})]\,dr. \tag{3.6b}$$

Differentiating Eq. (3.6a) and subtracting Eq. (3.6b), we obtain

$$\lambda(\tilde{\rho})\zeta_{,\rho^s}(\tilde{\rho}) = \rho^0_{,\rho^s}(\tilde{\rho})a(\rho^0(\tilde{\rho}))\exp[-\varphi(\rho^0(\tilde{\rho}),\tilde{\rho})], \quad s\in\{1,\ldots,k\}. \tag{3.7}$$

In order that the conditions (3.7) on the surface Γ should be consistent, the following conditions must be satisfied:

$$\frac{\lambda(\tilde{\rho})\cdot\zeta_{,\rho^1}(\tilde{\rho})}{\rho^0_{,\rho^1}(\tilde{\rho})} = \cdots = \frac{\lambda(\tilde{\rho})\cdot\zeta_{,\rho^k}(\tilde{\rho})}{\rho^0_{,\rho^k}(\tilde{\rho})},$$

that is,

$$\lambda(\tilde{\rho})\cdot d\zeta(\tilde{\rho}) \sim d\rho^0(\tilde{\rho}). \tag{3.8}$$

Hence taking into account that $d\lambda_1\wedge\cdots\wedge d\lambda_n \neq 0$, we have $\Sigma_{\mu=1}^m d\lambda_\mu \wedge d\zeta^\mu = 0$, so the function $\zeta^\mu(\tilde{\rho})$ must be of the form

$$\zeta^\mu = \frac{\partial W(\lambda_1,\ldots,\lambda_n)}{\partial \lambda_\mu}. \tag{3.9}$$

For such a form of the function $\zeta(\tilde{\rho})$ the regularity conditions on the surface Γ resolve themselves into

$$\operatorname{rank}\left\|\frac{\partial^2 W}{\partial\lambda_\mu\partial\lambda_\nu}\cdot\frac{\partial\lambda_\nu}{\partial\rho^t}\right\| = k.$$

In front of the assumption that $\operatorname{rank}\|\partial\lambda_\nu/\partial\rho^t\| = k$, the above condition is satisfied if we require that $\operatorname{Hess}(W)\neq 0$.

Inserting (3.9) into Eq. (3.8), we get

$$\lambda(\tilde{\rho})d\zeta = d\left(\frac{\partial W}{\partial\lambda_\mu}\lambda_\mu - W\right) \sim d\rho^0,$$

that is,

$$\rho^0 = \Psi\left(\frac{\partial W}{\partial\lambda_\mu}\lambda_\mu - W\right),$$

where Ψ is an arbitrary monotonic function of one variable. With these assumptions, Eqs. (3.7) take the form

$$a\left(\Psi\left(\frac{\partial W}{\partial\lambda_\mu}\lambda_\mu - W\right)\right) = \frac{\exp[\varphi(\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W),\tilde{\rho})]}{\dot{\Psi}((\partial W/\partial\lambda_\mu)\lambda_\mu - W)}. \tag{3.10}$$

Inserting (3.10) into Eq. (3.6a), we have

$$\Phi(\tilde{\rho}) = \lambda(\tilde{\rho})\cdot\zeta(\tilde{\rho}) - \int_0^{\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W)} a(r)\exp[-\varphi(r,\tilde{\rho})]\,dr. \tag{3.11}$$

Inserting (3.11) into the conditions (2.20) and using Eqs. (3.6) and (3.11), we obtain additional conditions for the functions $W(\cdot)$ and $\Psi(\cdot)$, i.e.,

$$\begin{aligned}
&P_{st}\int_0^{\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W)}[1-a(r)]\frac{\partial\varphi(r,\tilde{\rho})}{\partial\rho^t}\exp[-\varphi(r,\tilde{\rho})]\,dr\\
&\quad + P_{ts}\int_0^{\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W)}[1-a(r)]\frac{\partial\varphi(r,\tilde{\rho})}{\partial\rho^s}\\
&\quad\times\exp[-\varphi(r,\tilde{\rho})]\,dr\\
&\quad + Q_{st}\int_0^{\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W)}[a(r)-1]\\
&\quad\times\exp[-\varphi(r,\tilde{\rho})]\,dr\\
&\quad + \int_0^{\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W)}[a(r)-1]\\
&\quad\times\left(\frac{\partial^2\varphi(r,\tilde{\rho})}{\partial\rho^t\rho^s} - \frac{\partial\varphi(r,\tilde{\rho})}{\partial\rho^t}\cdot\frac{\partial\varphi(r,\tilde{\rho})}{\partial\rho^s}\right)dr\\
&\quad + \frac{\partial^2 W}{\partial\lambda_\mu\partial\lambda_\nu}\cdot\frac{\partial\lambda_\nu}{\partial\rho^t}\\
&\quad\times\left(\frac{\partial\varphi}{\partial\rho^s}\lambda_\mu + \frac{\partial\lambda_\mu}{\partial\rho^s}\right) = 0, \quad s\neq t\in\{1,\ldots,k\}.
\end{aligned} \tag{3.12}$$

Thus we can formulate the following.

Theorem 3: Let be given a regular k-dimensional surface Γ of the form

$$\Gamma = \{x = \zeta(\tilde{\rho})\}, \quad \tilde{\rho} := (\rho^1,\ldots,\rho^k).$$

Then the initial data for the system (3.1),

$$\tilde{R}(\zeta(\tilde{\rho})) = \tilde{\rho}, \quad R^0(\zeta(\tilde{\rho})) = \rho^0(\tilde{\rho})$$

(where ρ^0 is a given function on the surface Γ), are consistent, if the functions $\zeta(\tilde{\rho})$ and $\rho(\tilde{\rho})$ have the form

$$\zeta^\mu(\tilde{\rho}) = \frac{\partial W(\lambda(\tilde{\rho}))}{\partial\lambda_\mu}, \quad \rho^0(\tilde{\rho}) = \Psi\left(\frac{\partial W}{\partial\lambda_\mu}\lambda_\mu - W\right),$$

where W is a certain function determined on an open subset $\mathscr{E}^*$ and $\Psi(\cdot)$ is a monotonic function of one variable.

The solution of the initial problem with initial data (3.5)

for the system (3.1) has the implicit form

$$\lambda(\tilde{R})\cdot x = \Phi(\tilde{R}) + \int_0^{R^0} a(r)\exp[-\varphi(r,\tilde{R})]\,dr,$$

$$\frac{\partial\lambda(\tilde{R})}{\partial R^s}\cdot x = \frac{\partial\Phi(\tilde{R})}{\partial R^s} - \int_0^{R^0} a(r)\frac{\partial\varphi(r,\tilde{R})}{\partial R^s} \times \exp[-\varphi(r,\tilde{R})]\,dr,$$

where

$$a\left(\Psi\left(\frac{\partial W}{\partial\lambda_\mu}\lambda_\mu - W\right)\right) = \frac{\exp[\varphi(\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W),\tilde{R})]}{\dot{\Psi}((\partial W/\partial\lambda_\mu)\lambda_\mu - W)},$$

$$\Phi(\tilde{R}) = \lambda(\tilde{R})\cdot\xi(\tilde{R}) - \int_0^{\Psi((\partial W/\partial\lambda_\mu)\lambda_\mu - W)} a(r)\exp[-\varphi(r,\tilde{R})]\,dr,$$

while the functions W and Ψ satisfy additional conditions (3.12).

Let us consider now the special case when $k = 1$. Then the compatibility conditions (2.11) do not exist. In this case the initial data are given on a certain curve $L \subset \mathscr{E}$. Let us assume that along this curve there is given a value $R|_L = (R^0|_L, R^1|_L)$. Let us also assume that the functions $R^0|_L$ and $R^1|_L$ are invertible. Then as a parameter of this curve we can choose the value R^1, i.e.,

$$L = \{x = \eta^1(r^1)\}, \quad R\circ\eta^1(r^1) = (\rho^0(r^1), r^1)$$

or the value R^0, i.e.,

$$L = \{x = \eta^0(r^0)\}, \quad R\circ\eta^0(r^0) = (r^0, \rho^1(r^0)).$$

Then the functions $r^0 \to \rho^1(r^0)$ and $r^1 \to \rho^0(r^1)$ are inverses one to another. Moreover, by virtue of the identity $\eta^1(r^1) = \eta^0(\rho^0(r^1))$ we have[12]

$$\dot{\eta}^1(r^1) = \dot{\eta}^0(r^0)\dot{\rho}^0(r^1). \tag{3.13}$$

Inserting this into Eqs. (3.2) and (3.4), we get

$$G^0(R) = \lambda(r^1)\eta^1(r^1) = \Phi(r^1) + \int_0^{\rho^0(r^1)} a(r)\exp[-\varphi(r,r^1)]\,dr, \tag{3.14a}$$

$$G^1(R) = \dot{\lambda}(r^1)\eta^1(r^1) = \Phi(r^1) - \int_0^{\rho^0(r^1)} a(r)\frac{\partial\varphi(r,r^1)}{\partial r^1}\exp[-\varphi(r,r^1)]\,dr. \tag{3.14b}$$

Differentiating Eq. (31.4a) and subtracting Eq. (3.14b), we get

$$\lambda(r^1)\dot{\eta}^1(r^1) = \dot{\rho}^0(r^1)a(\rho^0(r^1))\exp[-\varphi(\rho^0(r^1),r^1)]. \tag{3.15}$$

Taking Eq. (3.13) into account, it follows that

$$a(r^0) = \lambda(\rho^1(r^0))\dot{\eta}^0(r^0)\exp[\varphi(r^0,\rho^1(r^0))] \tag{3.16}$$

and after inserting (3.16) into Eq. (3.14a) we have

$$\Phi(r^1) = \lambda(r^1)\eta^1(r^1) - \int_0^{\rho^0(r^1)} \lambda(\rho^1(r))\dot{\eta}^0(r) \times\exp[\varphi(r,\rho^1(r)) - \varphi(r,r^1)]\,dr. \tag{3.17}$$

Thus we get a theorem.

Theorem 4: If on some curve $L \subset \mathscr{E}$ the Cauchy conditions $R^0|_L$ and $R^1|_L$ are given for Eq. (3.1) in such a way that

(1) function $R^0|_L$ and $R^1|_L$ are strongly monotonic,

(2) for $x \in L$ a vector tangent to the curve L does not belong to the annihilator of the forms $\lambda^0(R(x))$ and $\lambda^1(R(x))$,

then in some neighborhood of the cuve L the Cauchy problem has (locally) exactly one solution.

Proof: It is enough to show that for functions G^0 and G^1 defined by formulae (3.14) and (3.17) the system (3.2) can be resolved in the neighborhood of an arbitrary point x on the curve L. Condition of local resolvability has the form

$$0 \neq \det\begin{vmatrix} \frac{\partial G^0}{\partial r^0}, & \frac{\partial G^0}{\partial r^1} - \dot{\lambda}x \\ \frac{\partial G^1}{\partial r^0}, & \frac{\partial G^1}{\partial r^1} - \ddot{\lambda}x \end{vmatrix} = \det\begin{vmatrix} \frac{\partial G^0}{\partial r^0}, & 0 \\ \frac{\partial G^1}{\partial r^0}, & \frac{\partial G^1}{\partial r^1} - \ddot{\lambda}x \end{vmatrix} = \frac{\partial G^0}{\partial r^0}\left(\frac{\partial G^1}{\partial r^1} - \ddot{\lambda}x\right),$$

where $x = \eta^0(r^0) \in \Gamma$. We have to prove, that $\partial G^0/\partial r^0 \neq 0$ and $\partial G^1/\partial r^1 - \ddot{\lambda}x \neq 0$. By virtue of (3.14a) and (3.16) we have

$$\frac{\partial G^0}{\partial r^0} = a(r^0)\exp[-\varphi(r^0,r^1)] = \lambda(r^1)\dot{\eta}^0(r^0)\exp[-\varphi(r^0,r^1) + \varphi(r^0,r^1)] = \lambda(r^1)\dot{\eta}^0(r^0) \neq 0 \tag{3.18}$$

because by assumption (2) the tangent vector $\dot{\eta}^0(r^0)$ does not belong to the annihilator of $\lambda(r^1)$. Similarly, utilizing Eqs. (3.3), (3.14b), and (3.18), we have

$$\frac{\partial G^1}{\partial r^1} - \ddot{\lambda}x = \frac{d}{dr^1}[G^1(\rho^0(r^1),r^1)] - \frac{\partial G^1}{\partial r^0}\dot{\rho}^0 - \ddot{\lambda}x = \dot{\lambda}\cdot\dot{\eta}^1 + \frac{\partial\varphi}{\partial r^1}\frac{\partial G^0}{\partial r^0} = \left(\frac{\partial\varphi}{\partial r^1}\lambda + \dot{\lambda}\right)\dot{\eta}^1 \neq 0$$

because by assumption (2) the tangent vector $\dot{\eta}^1$ does not vanish on $\lambda^1 = (\partial\varphi/\partial r^1)\lambda + \dot{\lambda}$. Q.E.D.

A. The case when all $\alpha^s \neq 0$

In the case of nonhomogeneous system (2.13), according to the consideration in Sec. II, we have

$$dR^0 = \lambda(\tilde{R})\exp[\varphi(R)], \qquad dR^s = \xi^s\left(\frac{\partial\varphi(R)}{\partial R^s}\lambda(\tilde{R}) + \frac{\partial\lambda(\tilde{R})}{\partial R^s}\right), \tag{3.19}$$

and the solution is also determined by formulae (3.2), (3.4), and (3.12) with additional restriction

$$a\left(\Psi\left(\frac{\delta W}{\partial\lambda_\mu}\lambda_\mu - W\right)\right) \equiv 1 \tag{3.20}$$

on the arbitrary functions. Inserting (3.20) into Eqs. (3.10) and (3.12), we obtain the relations

$$\exp\left[\varphi\left(\Psi\left(\frac{\delta W}{\partial\lambda_\mu}\lambda_\mu - W\right),\tilde{\rho}\right)\right] \equiv \dot{\Psi}\left(\frac{\delta W}{\partial\lambda_\mu}\lambda_\mu - W\right), \tag{3.21}$$

$$\frac{\partial^2 W}{\partial\lambda_\mu\partial\lambda_\nu}\frac{\partial\lambda_\nu}{\partial R^t}\left(\frac{\partial\varphi}{\partial R^s}\lambda_\mu \div \frac{\partial\lambda_\mu}{\partial R^s}\right) = 0, \tag{3.22}$$

which are the restrictions on the functions Ψ and W. Thus we have:

Theorem 5: Let be given a regular k-dimensional surface Γ of the form

$$\Gamma = \{ x = \zeta(\tilde{\rho}) \}, \quad \tilde{\rho} := (\rho^1, \ldots, \rho^k).$$

Then the initial data for the system (3.19),

$$\tilde{R}(\zeta(\tilde{\rho})) = \tilde{\rho}, \quad R^0(\zeta(\tilde{\rho})) = \rho^0(\tilde{\rho}) \tag{3.23}$$

(where ρ^0 is the given function on the surface Γ), are consistent, if the functions $\zeta(\tilde{\rho})$ and $\rho^0(\tilde{\rho})$ have the form

$$\zeta^\mu(\tilde{\rho}) = \frac{\partial W(\lambda(\tilde{\rho}))}{\partial \lambda_\mu}, \quad \rho^0(\tilde{\rho}) = \Psi\left(\frac{\partial W}{\partial \lambda_\mu}\lambda_\mu - W\right),$$

where W is a certain function determined on an open subset $\mathscr{E}^*$ and $\Psi(\cdot)$ is a monotonic function of one variable.

If the conditions (3.21) and (3.22) are satisfied, then the solution of the initial problem [with initial data (3.23)] for the system (3.19) has the implicit form

$$\lambda(\tilde{\rho})\zeta(\tilde{\rho}) = \Phi(\tilde{\rho}) + \int_0^{\rho^0(\tilde{\rho})} \exp[-\varphi(r,\tilde{\rho})]\, dr,$$

$$\frac{\partial \lambda(\tilde{\rho})}{\partial \rho^s}\zeta(\tilde{\rho}) = \frac{\partial \Phi(\tilde{\rho})}{\partial \rho^s} - \int_0^{\rho^0(\tilde{\rho})} \frac{\partial \varphi(r,\tilde{\rho})}{\partial \rho^s} \exp[-\varphi(r,\tilde{\rho})]\, dr.$$

Remark 4: When $k = 1$, the solution of the system (3.19) is also determined by formulae (3.2) and (3.4) with additional restriction $a(r^0) \equiv 1$ on the arbitrary functions. From Eq. (3.16) we see that in this case along the curve L we can give only one function, for example, $R^0|_L$ and then the other one may be computed from the restriction $a(r^0) \equiv 1$. So we have:

Theorem 6: Let along some curve L be given the function $R^0|_L$. Let us assume that:

(1) the function $R^0|_L$ is monotonic,

(2) equation $\lambda^0(r^0, r^1)\eta^0(r^0) = 1$ [where $x = \eta^0(r^0)$ is the equation of the curve L parametrized by $R^0|_L$] allows us along the curve L, to determine uniquely the value $r^1 = R^1|_L$,

(3) the values $R^0|_L$ and $R^1|_L$ determined in this way satisfy the transversality condition[13] with respect to the form $\lambda^1(R|_L)$, then the Cauchy problem

$$R^0(\eta^0(r^0)) = r^0$$

has, in some neighborhood of the curve L, exactly one solution.

B. The case when all $\alpha^s = 0$

Let along some regular k-dimensional surface

$$\Gamma = \{ x = \zeta(\tilde{\rho}) \}, \quad \tilde{\rho} = (\rho^1, \ldots, \rho^k)$$

in space $\mathscr{E}$ be given a value $R|_\Gamma$, i.e., the initial data have the form

$$R^s(\zeta(\tilde{\rho})) = \rho^s.$$

Moreover, let be given a value

$$R^0(\zeta(\tilde{\rho}_0)) = \rho_0^0 \in \mathbb{R}^1.$$

Then it follows from Eq. (2.26a) that

$$R^0(\zeta(\tilde{\rho})) = \chi^{-1}(C\cdot\zeta(\tilde{\rho}) + \rho_0^0 - C\cdot\zeta(\tilde{\rho}_0)) =: \rho^0(\tilde{\rho}). \tag{3.24}$$

From Eq. (2.26b) we can determine the value of the functions Ψ^s, i.e.,

$$\Psi^s(\tilde{\rho}) = \int_0^{(\zeta(\tilde{\rho}) + \rho_0^0 - \zeta(\tilde{\rho}_0))} \kappa^s(\chi^{-1}(r), \tilde{\rho})dr + A^s(\tilde{\rho})\zeta(\tilde{\rho}). \tag{3.25}$$

We look for the conditions of the local resolvability of the system (2.26). As Eq. (2.26a) always allows us to find the value of $R^0 = R^0(x)$, so these conditions reduce to the nonsingularity of the matrix

$$M_t^s := \frac{\partial}{\partial R^t}\left\{\int_0^{C\cdot x + C} \kappa^s(\chi^{-1}(r), \tilde{R})dr + A^s(\tilde{R})x - \Psi^s(\tilde{R})\right\} = \int_0^{C\cdot x + C_0} \frac{\partial \kappa^s(\chi^{-1}(r), \tilde{R})}{\partial R^t} dr + \frac{\partial A^s(\tilde{R})\cdot x}{\partial R^t} - \frac{\partial \Psi^s(\tilde{R})}{\partial R^t},$$

where $s, t \in \{1, \ldots, k\}$. Differentiating (3.25), we have

$$\frac{\partial \Psi^s(\tilde{\rho})}{\partial \rho^t} = C\frac{\partial \zeta(\tilde{\rho})}{\partial \rho^t}\kappa^s(\rho^0(\rho), \tilde{\rho}) + \int_0^{C\cdot x + C_0} \frac{\partial \kappa^s(\chi^{-1}(r), \tilde{\rho})}{\partial \rho^t} dr + \frac{\partial A^s(\tilde{\rho})}{\partial \rho^t}\cdot\zeta(\tilde{\rho}) + A^s(\tilde{\rho})\frac{\partial \zeta(\tilde{\rho})}{\partial \rho^t}. \tag{3.26}$$

So the matrix M_t^s takes the form

$$M_t^s = C\cdot\frac{\partial \zeta(\tilde{\rho})}{\partial \rho^t}\kappa^s(\rho^0(\tilde{\rho}), \tilde{\rho}) + A^s(\tilde{\rho})\cdot\frac{\partial \zeta(\tilde{\rho})}{\partial \rho^t} = \lambda^s(\rho^0(\tilde{\rho}), \tilde{\rho})\frac{\partial \zeta(\tilde{\rho})}{\partial \rho^t}. \tag{3.27}$$

Thus we have:

Theorem 7: Let us assume that:

(1) along the regular k-dimensional surface $\Gamma \subset \mathscr{E}$ are given the values $R^s|_\Gamma$ being the monotonic functions of their arguments,

(2) for the value $R^0|_\Gamma$ determined from the formula (3.24) any tangent vectors to the surface Γ do not belong to the annihilator of the forms $\lambda^s(R)|_\Gamma$,

(3) the conditions (2.24) are satisfied.

Then the Cauchy problem has exactly one solution in some neighborhood of the surface Γ.

IV. FUNCTORIAL PROPERTIES OF THE SYSTEM OF EQUATIONS DETERMINING RIEMANN INVARIANTS

Let $A: \overline{\mathscr{E}} \to \mathscr{E}$ be the linear operator, and $A^*: \mathscr{E}^* \to \overline{\mathscr{E}}^*$ be the dual operator. Then we can connect the system

$$dR^s = \xi^s\lambda^s(R), \quad R = (R^s): \mathscr{E} \to H, \quad s \in \{0, 1, \ldots, k\}, \tag{4.1}$$

$$\xi^0 = \begin{cases} 0 & \text{for homogeneous system (2.6)} \\ 1 & \text{for nonhomogeneous system (2.6)} \end{cases}, \qquad \dim \mathscr{E} = n + 1,$$

where $\lambda^s: H \to \mathscr{E}$, with the system of equations

$$d\overline{R}^s = \xi^s\overline{\lambda}^s(\overline{R}), \quad \overline{R} = (\overline{R}^s): \overline{\mathscr{E}} \to H, \quad \dim \overline{\mathscr{E}} = n + 1, \tag{4.2}$$

where $\overline{\lambda}^s: A^* \circ \lambda^s: H \to \overline{\mathscr{E}}^*$, $s \in \{0, 1, \ldots, k\}$.

Let us investigate the relation between the solutions of the system (4.1) and (4.2). We notice that if $R: \mathscr{E} \to H$ satisfies

(4.1), i.e., $dR^s(x) = \xi^s \lambda^s(R(x))$, then for $\overline{R} := R \circ A$ we have

$$d\overline{R}^s(y) = d(R^s \circ A)(y) = dR^s(Ay) \circ A = A^* \, dR^s(Ay)$$
$$= A^*[\xi^s \lambda^s(R(Ay))] = \xi^s A^* \overline{\lambda}^s(\overline{R}(y)).$$

So $\overline{R} = R \circ A$ satisfies the system (4.2).

Now we will show that if covectors $\overline{\lambda}^s = A^* \circ \lambda^s$ are linearly independent, then all solutions of the system (4.2) can be obtained in this way.

It is enough to check that every solution $\overline{R}: \mathscr{E} \to H$ of the system (4.2) is of the form $\overline{R} = R \circ A$, where $R: \mathscr{E} \to H$ satisfies the system (4.1). Therefore, we have to prove that the following formulation is valid:

$$T(x) := \overline{R}(y) \quad \text{if } x = Ay,$$

that is $Ay_1 = Ay_2$ implies $\overline{R}(y_1) = \overline{R}(y_2)$.

Indeed, we have

$$\overline{\lambda}^s(\overline{R})(y_2 - y_1) = \lambda^s(\overline{R})A(y_2 - y_1)$$

so that

$$\overline{R}(y_2) - \overline{R}(y_1) = \int_0^1 \frac{d}{dt} \overline{R}(y_1 + t(y_2 - y_1))\, dt$$
$$= \int_0^1 d\overline{R}(y_1 + t(y_2 - y_1))(y_2 - y_1) dt$$
$$= \int_0^1 \sum_{s=1}^{p} \overline{\lambda}^s(\overline{R}(y_1 + t(y_2 - y_1)))$$
$$\times (y_2 - y_1)\, dt = 0.$$

Thus we have defined the mapping $T: \mathscr{E}_0 \to H$, where $\mathscr{E}_0 := \operatorname{im} A \in \mathscr{E}$.

If A is a surjection, then $R := T$ is defined on the whole space $\mathscr{E}$. Let us show that R satisfies Eq. (4.1). For $y \in \overline{\mathscr{E}}$ and $x = Ay$, we have

$$d\overline{R}^s(y) = \xi^s \overline{\lambda}^s(R(x))$$

so $A^* \circ [dR^s(x) - \xi^s \lambda^s(R(x))] = 0$. From injectivity it follows that (4.1) holds.

We will give another proof of this theorem under the assumption that $\overline{\lambda}^s = A^* \circ \lambda^s$ are linearly independent and that the system (4.1) is in involution. For the homogeneous system (4.1) the forms $\overline{\lambda}^s(\overline{R})$ and $\lambda^s(R)$ satisfy the same conditions of involution[14,15] (see Paper I, Sec. III), i.e.,

$$\overline{\lambda}^s{}_{,\overline{R}^r} = \alpha^s_r \overline{\lambda}^s + \beta^s_r \overline{\lambda}^r, \quad \lambda^s{}_{,\overline{R}^r} = \alpha^s_r \lambda^s + \beta^s_r \lambda^r, \quad r \neq s.$$

As we know, the solution $\overline{R}$ of the system (4.2) can be expressed in the implicit form

$$\Psi^s(\overline{R}) = \overline{\lambda}^s(\overline{R}) \cdot y = \lambda^s(\overline{R}) \cdot Ay, \tag{4.3}$$

where the functions Ψ^s are solutions of the system

$$\Psi^s{}_{,\overline{R}^r} = \alpha^s_r \Psi^s + \beta^s_r \Psi^r, \quad r \neq s.$$

On the other hand, resolving the system

$$\Psi^s(R) = \lambda^s_\mu(R) x^\mu, \tag{4.4}$$

with respect to the variable R, we get the solution of the system (4.1). Inserting $x = Ay$ into (4.4) and utilizing (4.3), we find $R(x) = \overline{R}(y)$. That means that $\overline{R} = R \circ A$, and this ends the proof.

For the nonhomogeneous system (2.6) the forms $\overline{\lambda}^s(\overline{R})$ and $\lambda^s(R)$ satisfy the same conditions of involution (2.7), i.e.,

$$\overline{\lambda}^0{}_{,\overline{R}^\sigma} = \alpha_\sigma \overline{\lambda}_\sigma, \quad \lambda^0{}_{,\overline{R}^\sigma} = \alpha_\sigma \lambda^\sigma,$$
$$\overline{\lambda}^t{}_{,\overline{R}^\sigma} = \beta^t_\sigma \overline{\lambda}^t + \zeta^t_\sigma \overline{\lambda}^\sigma, \quad \lambda^t{}_{,\overline{R}^\sigma} = \beta^t_\sigma \lambda^t + \zeta^t_\sigma \lambda^\sigma, \tag{4.5}$$
$$\sigma \in \{0,1,\ldots,k\}, \quad t \in \{1,\ldots,k\}, \quad \sigma \neq t.$$

According to the consideration in Sec. II, we can determine the form of covectors $\overline{\lambda}^s(\overline{R})$ and $\lambda^s(R)$ for which the compatibility conditions (4.5) are automatically satisfied. So we have

$$\overline{\lambda}^0(\overline{R}) = \overline{\lambda}(\tilde{\overline{R}}) \exp[\varphi(\overline{R})],$$
$$\overline{\lambda}^t(\overline{R}) \sim [\overline{\lambda}(\tilde{\overline{R}}) \varphi_{,\overline{R}^t}(\overline{R}) + \overline{\lambda}_{,\overline{R}^t}(\tilde{\overline{R}})],$$
$$\lambda_{,\overline{R}^\sigma \overline{R}^t}(\tilde{\overline{R}}) = P_{st} \lambda_{,\overline{R}^t}(\tilde{\overline{R}}) + P_{ts} \overline{\lambda}_{,\overline{R}^s}(\tilde{\overline{R}}) + Q_{st} \overline{\lambda}(\tilde{\overline{R}}), \quad s \neq t,$$

and

$$\lambda^0(R) = \lambda(\tilde{R}) \exp[\varphi(R)],$$
$$\lambda^t(R) \sim [\lambda(\tilde{R}) \varphi_{,R^t}(R) + \lambda_{,R^t}(\tilde{R})],$$
$$\lambda_{,R^s R^t}(\tilde{R}) = P_{st} \lambda_{,R^t}(\tilde{R}) + P_{ts} \lambda_{,R^s}(\tilde{R}) + Q_{st} \lambda(\tilde{R}), \quad s \neq t.$$

As we know, the solution $\overline{R}$ of the system (4.2) can be expressed in the implicit form

$$\Psi^0(\overline{R}) = \overline{\lambda}(\tilde{\overline{R}}) \cdot y = \lambda(\tilde{\overline{R}}) \cdot Ay,$$
$$\Psi^t(\overline{R}) = \Psi^0{}_{,\overline{R}^t}(\tilde{\overline{R}}) = \overline{\lambda}_{,\overline{R}^t}(\tilde{\overline{R}}) \cdot y = \lambda_{,\overline{R}^t}(\tilde{\overline{R}}) \cdot Ay,$$

where the function Ψ^0 is a solution of the system

$$\Psi^0{}_{,\overline{R}^s \overline{R}^t}(\overline{R}) = P_{ts} \Psi^0{}_{,\overline{R}^s}(\overline{R}) + P_{st} \Psi^0{}_{,\overline{R}^t}(\overline{R}) + Q_{ts} \Psi^0(\overline{R}),$$
$$t \neq s \in \{1,\ldots,k\}. \tag{4.6}$$

On the other hand, resolving the system

$$\Psi^0(R) = \lambda(\tilde{R}) \cdot x, \quad \Psi^t(R) = \Psi^0{}_{,R^t}(R) = \lambda_{,R^t}(\tilde{R}) \cdot x, \tag{4.7}$$

with respect to the variable R we get the solution of the system (4.1). Inserting $x = A \cdot y$ into Eqs. (4.7) and utilizing Eqs. (4.6), we find $R(x) = \overline{R}(y)$. That means that $\overline{R} = R \circ A$ and this ends the proof.

Examples: Let $\overline{\mathscr{E}} \subset \mathscr{E}$, be such a subspace that $\lambda^s(R)|_{\overline{\mathscr{E}}}$ are linearly independent. It is easy to see that if $A: \overline{\mathscr{E}} \to \mathscr{E}$ is immersion, then $A^* \lambda^s(R) = \lambda^s(R)|_{\overline{\mathscr{E}}}$. In this case the above solutions indicate that the projection $\overline{R} := R|_{\mathscr{E}}$ of the solution R of the system (4.1) in the whole space $\mathscr{E}$ determines the solution of the system in the subspace $\overline{\mathscr{E}}$, and, conversely, every solution R of the system (4.1) can be uniquely obtained from the solution $\overline{H}$ of the system (4.2) in the subspace $\overline{\mathscr{E}} \subset \mathscr{E}$. We can give the following interpretation to the extension of $\overline{R}$ to the R:

Let us define at every point $x_0 \in \mathscr{E}$ a hyperplane

$$H(x_0) := x_0 + An\{\lambda^1(\overline{R}(x_0)), \ldots, \lambda^p(\overline{R}(x_0))\}.$$

We see that $\overline{R}$ is constant on the intersection of such hyperplane with $\mathscr{E}$. Additionally,

$$\overline{\mathscr{E}} + An\{\lambda^1(\overline{R}(x_0)), \ldots, \lambda^p(\overline{R}(x_0))\} = \mathscr{E}$$

because in the case

$$\overline{\mathscr{E}} + An\{\lambda^1(\overline{R}(x)), \ldots, \lambda^p(\overline{R}(x))\} \neq \mathscr{E}$$

there would exist a nonzero element $\lambda \in \mathscr{E}^*$ which would vanish on $\overline{\mathscr{E}}$ and

$$AnAn\{\lambda^1(\overline{R}(x_0)), \ldots, \lambda^p(\overline{R}(x_0))\}$$
$$= \{\lambda^1(\overline{R}(x_0)), \ldots, \lambda^p(\overline{R}(x_0))\},$$

which would be in contradiction with the assumption that $\lambda^s(R)|_{\overline{\mathscr{E}}}$ are linearly independent. Thus the hyperplanes

$H(x_0)$, $x_0 \in \mathscr{E}$, cover some neighborhood of $\mathscr{E}$ in the space $\mathscr{E}$. The solution R of the system (4.1) can be determined as $R(x) := R(x_0)$ for $x \in H(x_0)$.

Suppose that some $\overline{\lambda}^s : H \to \overline{\mathscr{E}}^*$ have common constant kernel $\mathscr{E}_0 \in \overline{\mathscr{E}}$, i.e., $\lambda^s(R)|_{\mathscr{E}_0} = 0$ for $R \in H$. In this case for

$$A: \overline{\mathscr{E}} \to \overline{\mathscr{E}}/\mathscr{E} =: \mathscr{E}$$

the above considerations give us a procedure for the reduction of the system (4.2) to the smaller space $\mathscr{E} = \overline{\mathscr{E}}/\mathscr{E}$. Hence, the solutions $\overline{R}: \overline{\mathscr{E}} \to H$ are constant along the fiber layers A and "functorial mapping" $R: \mathscr{E} \to H$ is the solution of the Eq. (4.1).

V. PHYSICAL INTERPRETATION OF THE NONLINEAR SUPERPOSITION OF THE SIMPLE WAVES IN NONHOMOGENEOUS SYSTEMS

Now we show that the solutions of the system of Pfaff forms (2.1) can be interpreted as the interaction of many simple waves propagating on the simple state. For the sake of simplicity this statement will be illustrated on selected examples. Let us consider a system of nonhomogeneous quasilinear partial differential equations with two independent variables (let it be time t and one space variable x):

$$u_t + A(u)u_x = B(u), \tag{5.1}$$
$$(t,x) \in \mathscr{E} \subset R^2, \quad u(x) = (u^1(x),\ldots,u^l(x)) \in \mathscr{H} \subset R^l,$$

where $A = A^j_s(u)$ is a matrix of the dimensional $l \times l$, while $B(u) = (B^1(u),\ldots,B^l(u))$ is vector with l-components. Simple elements for homogeneous and nonhomogeneous equations (5.1) are determined by algebraic equations as follows:

$$(Iv_s - A)\gamma_s = 0, \quad \lambda^s = (v_{(s)} - 1), \quad s \in \{1,\ldots,k\}, \tag{5.2}$$
$$(I\lambda^0_1 + A\lambda^0_2)\gamma_0 = B, \quad \lambda^0 = (\lambda^0_1, \lambda^0_1).$$

Let us also consider the propagation of two simple waves on a simple state [$k = 2$ in Eqs. (2.1)]. By the appropriate normalization of the length of the vectors $\gamma_0,\gamma_1,\gamma_2$ we can make the commutators (2.2) vanish. Thus we can choose the parametrization of the surface

$$u = f(R), \quad R := (R^0, R^1, R^2) \tag{5.3}$$

tangent to the fields $\gamma_0,\gamma_1,\gamma_2$ in the following way:

$$\gamma_\sigma \simeq \frac{\partial f}{R^\sigma}, \quad \sigma = 0,1,2. \tag{5.4}$$

Hence $du(x) = \Sigma^2_{\sigma=0} u_{,R^\sigma} dR^\sigma$. From (2.1), under the assumption that $\gamma_0,\gamma_1,\gamma_2$ are linearly independent, we have

$$dR^s = \xi^s \lambda^s, \quad s = 1,2, \quad dR^0 = \lambda^0. \tag{5.5}$$

For such a case the system of Eqs. (5.5) can be written in terms of (only three now) new dependent variables. Eliminating the variables ξ^s in Eq. (5.5), we obtain

$$R^s{}_{,t} + v_{(s)}(R)R^s{}_{,x} = 0, \quad s = 1,2, \tag{5.6a}$$
$$R^0{}_{,t} = \lambda^0_1(R^0, R^1, R^2), \tag{5.6b}$$
$$R^0{}_{,x} = \lambda^0_2(R^0, R^1, R^2). \tag{5.6c}$$

Now we will show that the solution of the system (5.6) describes interaction of waves, and we will justify the name "simple wave on a simple state" for it.

It was proved in Refs. 16 and 17 that if the initial data are sufficiently small, then there exists such a time interval $[t_0,T]$, in whch the gradient catastrophy[18] for the solution $R^\sigma(t,x)$, $\sigma = 0,1,2$, of the system (5.6) does not occur, Since each of the functions $R^s(t,x)$, $s = 1,2$, is constant along the appropriate characteristics $C^{(s)}: dx/dt = v_s$ of the system (5.6), then if we choose in the space $\mathscr{E}$ the initial conditions for the function R^s such that the derivatives $R^s{}_{,x}$ will have compact and noninteracting supports

$$\operatorname{supp} R^s{}_{,x}(t_0,x) \subset [a_s,b_s], \tag{5.7}$$
$$\bigvee_{a_s,b_s \in R^s} \operatorname{supp} R^1{}_{,x}(t_0,x) \cap \operatorname{supp} R^2{}_{,x}(t_0,x) = \varnothing, \quad s = 1,2,$$

then, for arbitrary time $t_0 < t < T$, $\operatorname{supp} R^s{}_{,x}(t,x)$ is contained in the "stripe" between appropriate characteristics of the families $C^{(s)}$ passing through the ends of the interval $[a_s,b_s]$.

It was also proved[19] that if the initial data is sufficiently small, then in the interval $[t_0,T]$ the following condition can be satisfied:

$$\bigvee_{c>0} \bigwedge_{(t,x),(t,x') \in [t_0,T) \times R^1} v_1(R(t,x)) - v_2(R(t,x')) \geqslant c, \tag{5.8}$$

i.e., that every characteristic of the family $C^{(1)}$ has the tangent with inclination (measured with respect to the positive direction of the x axis) smaller than any characteristic of the family $C^{(2)}$.

It is evident that in such a case the stripes containing $\operatorname{supp} R^s{}_{,x}(t,x)$ divide the remaining part of the space $\mathscr{E}$ into the four disjoint regions (I-IV, respectively; see Fig. 1).

In the region $G: \mathscr{E} \setminus [\operatorname{supp} R^1{}_{,x}(t,\cdot) \cup \operatorname{supp} R^2{}_{,x}(t,\cdot)]$ the solution $R^0(t,x)$ of the system (5.6) is described by the simple state. In this region $R^s{}_{,x} = 0$ and the solution $R^0(t,x)$ satisifies Eqs. (5.6b) and (5.6c), with $R^s = R^s_0 = \text{const}$. From the condition of compatibility of this equation, we obtain

$$\lambda^0{}_{,R^0} \wedge \lambda^0 = 0,$$

which means that the direction of λ^0 does not depend on the variable R^0, so it is constant on G. Choosing the parametriza-

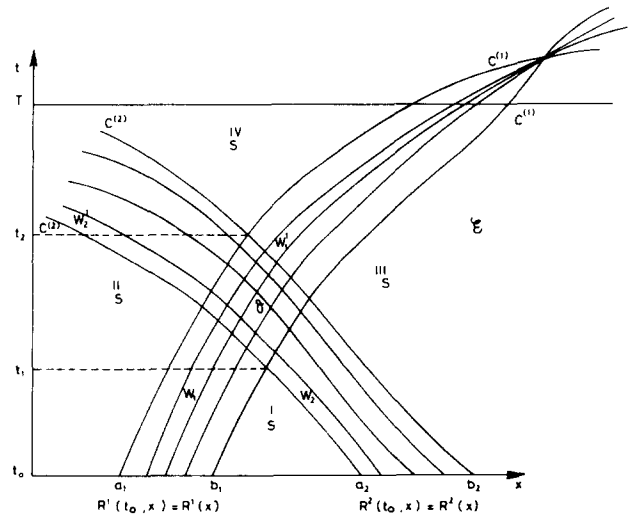

FIG. 1. The case of propagation of two simple waves on the simple state. S = simple state; W_1W_2 = simple waves propagating on the simple state S. If the characteristics of one family intersect, we choose a particular value of time T in order to exclude the possibility of gradient catastrophe.

tion of the curve $u = f(R^0, R^1, R^2)$ such that the covector λ^0 does not depend on the parameter R^0, we can express the solution on G in the form of the simple state (4.3) (see Paper I), i.e.,

$$u^j = f^j(R^0, R^1_0, R^2_0), \quad \text{where } \frac{df^j}{dR^0} = \gamma^j_0(f(R^0, R^1_0, R^2_0)),$$

$$j = 1,\ldots,l,$$

$$R^0 = \lambda^0_1 t + \lambda^0_2 x, \quad \lambda^0 = \text{constant vector}.$$

Now let t_1 and t_2 be the moments, in which $\text{supp}R^1_{,x}(t,x)$ and $\text{supp}R^2_{,x}(t,x)$ have only one common point. For times $t \in (t_0, t_1)$ we have $\text{supp}R^1_{,x}(t,x) \cap \text{supp}R^2_{,x}(t,x) = \emptyset$ so the solution $R^s(t,x)$ can be interpreted as the propagation of two separate (noninteracting) simple waves on the simple state. For the times $t \in [t_1, t_2]$ the characteristics of the families $C^{(1)}$ and $C^{(2)}$ containing $\text{supp}R^s_{,x}(t,x)$ cross each other, i.e., $\text{supp}R^1_{,x}(t,x) \cap \text{supp}R^2_{,x}(t,x) \neq \emptyset$. We interpret this as an interaction of two simple waves on the simple state. For times $t > t_2$, by virtue of the conditions (5.7) and (5.8), the "stripes" containing the supports of the simple waves separate again, i.e., we again have $\text{supp}R_{,x}(t,x) \cap \text{supp}R^2_{,x}(t,x) = \emptyset$, where $R^s_{,x} = -\xi^s$, $s = 1,2$. It means, that the solution $R^\sigma(t,x)$ decays in the exact way into the two simple waves propagating on the simple state.

We see that, under all the above assumptions, in the interaction of the two simple waves on the simple state, which can be described by Riemann invariants, the solution decays in the exact way into the two simple waves on the simple state, being of the same type as in the initial moment.

We have in this case the law of conservation of the number and type of the simple waves propagating on the simple state, so we can speak about elastic interaction of simple waves on the simple state.

In the case of more then two simple waves ($k > 2$) the interpretation is analogous but more complicated, because the region $\mathscr{D}$ is divided by supports of the functions $R^1,\ldots,R^k$ into the 2^k subsets.

In Refs. 6 and 19 it was proved that in the case of interaction of many simple waves described by Riemann invariants the number and the type of waves[20] is also conserved. It was also proved that such solutions, resulting from the interaction of many simple waves propagating on the simple state, decay in the exact way into simple waves on the state, of the same kind as those entering the interaction.

The possibility of the nonlinear superposition of simple waves on a simple state in the case that can not be described by Riemann invariants was discussed in Refs. 6 and 19. The conditions guaranteeing the existence of the solutions of the Pfaff system (2.1) with two independent variables take the form[19]

$$[\gamma_s, \gamma_r] = \sum_{p=1}^{k} C^p_{sr}\gamma_p, \qquad \bigwedge_{s,r \in \{1,\ldots,k\}}, \tag{5.9a}$$

$$[\gamma_s, \gamma_0] = \sum_{\sigma=1}^{k} C^\sigma_{s0}\gamma_\sigma,$$

$$(\lambda^0_{,\gamma_r} - \langle \mathring{\omega}, [\gamma_r, \gamma_0] \rangle \lambda^0) \wedge \lambda^r = 0,$$

$$\lambda^0 M \lambda^0_{,\gamma_0} = 0, \quad \lambda^\sigma \wedge \lambda^\alpha \neq 0 \tag{5.9b}$$

$$\text{for } \alpha \neq \sigma, \ \alpha,\sigma \in \{0,1,\ldots,k\},$$

where the covector $\mathring{\omega} \in T_u\mathscr{H}$ has the property $\langle \omega, \gamma_\sigma \rangle = \delta^0 r$ and the coefficients $C^\sigma_{\alpha\beta}$ are not necessarily constant.

The physical interpretation of this solution covers also the case of generation of waves as a result of their interaction on a simple state. Generation can take place for example when the commutators (5.9) of the vector fields γ_α and γ_β are linear combinations of these fields γ_α, γ_β, and also of other fields $\gamma_\rho,\ldots,\gamma_\sigma$. It means that the waves connected with these fields $\gamma_\rho,\ldots,\gamma_\sigma$ take part in the interaction. So as an effect of the interaction of two simple waves we obtain new waves (of another type than those at the initial moment). If these new waves do not vanish asymptotically for large t, then the effect of generation is permanent (in accordance with the accepted terminology it is "a nonelastic interaction").

VI. THE EXAMPLES OF APPLICATIONS FOR NONHOMOGENEOUS EQUATIONS OF MAGNETOHYDRODYNAMICS

We show as an illustration the example of exact solution obtained by means of the procedure described here (other examples of such solutions for the equations for gas and magnetohydrodynamic can be found in the papers[4,6,19,21]). Let us consdier the flow of ideal compressible conductive fluid placed in magnetic field in the presence of Coriolis forces $B = \rho\mathbf{\Omega} \times \mathbf{u}$. We assume that all unknown functions depend only on two independent variables (let them be time t and one space variable x). With the above assumptions the equations examined form the quasilinear hyperbolic system as follows[22–24]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u^1) = 0, \tag{6.1a}$$

$$\frac{\partial}{\partial t}\left(\frac{P}{\rho^\kappa}\right) + u^1 \frac{\partial}{\partial x}\left(\frac{p}{\rho^\kappa}\right) = 0, \tag{6.1b}$$

$$\frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^1}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{(H^2)^2 + (H^3)^2}{8\pi}\right) = 0, \tag{6.1c}$$

$$\frac{\partial u^2}{\partial t} + u^1 \frac{\partial u^2}{\partial x} - \frac{H^1_0}{4\pi\rho}\frac{\partial}{\partial x} H^2 = \rho\omega u^3, \tag{6.1d}$$

$$\frac{\partial u^3}{\partial t} + u^1 \frac{\partial u^3}{\partial x} - \frac{H^1_0}{4\pi\rho}\frac{\partial}{\partial x} H^3 = \rho\omega u^2, \tag{6.1e}$$

$$\frac{\partial H^2}{\partial t} + \frac{\partial}{\partial x}(u^1 H^2_0 - u^2 H^1_0) = 0, \tag{6.1f}$$

$$\frac{\partial H^3}{\partial t} + \frac{\partial}{\partial x}(u^1 H^3 - u^3 H^1_0) = 0, \quad \text{where } \mathbf{\Omega} = (\omega,0,0), \tag{6.1g}$$

where we used the following notations: ρ = the density of the fluid, p = the pressure of the fluid, $\mathbf{u}$ = the vector field of the fluid velocity, $\mathbf{H}$ = the vector of the magnetic field, $\mathbf{\Omega}$ = the angular velocity of the fluid and κ = the polytropic exponent. We can assume, without losing generality, that the coefficient of the magnetic inductive capacity equals the unity: $\mu = 1$. We recall here that in the case of two independent variables t and x the first component of the magnetic field H^1_0 has a constant value. So in this case the system (6.1) reduces to seven equations for seven unknown functions. It

TABLE I. Homogeneous simple elements. We introduce the notation $\delta = v_i - u^1$. The function δ possesses the physical sense. It corresponds to the velocity of the propagation of the disturbance relative to the fluid. v_i is the local velocity of the ith eigenvalue wave.

$$\delta_{\pm} = \frac{1}{2}\left[\left(\frac{\kappa p}{\rho} + \frac{H^1}{\sqrt{4\pi\rho}}\right)^2 + \frac{\tilde{H}^2}{4\pi\rho}\right]^{1/2} \pm \left[\left(\frac{\kappa p}{\rho} - \frac{H^1}{\sqrt{4\pi\rho}}\right)^2 + \frac{\tilde{H}}{4\pi\rho}\right]^{1/2},$$

$$\Delta_{\pm} = \delta^2_{\pm} - \frac{(H^1)^2}{4\pi\rho}, \quad \hat{H}^i = \frac{H^i}{4\pi\rho}, \quad i = 1, 2, 3.$$

No.	Notation		sgn. $\Delta \pm$ $\delta \pm$	Eigenvalues v_s	Eigenvector γ_s	Name of the wave corresponding to the eigenvector γ_s
1	E	$+1$		$v_1 = u^1$	$(1, 0, 0, \tilde{0}, \tilde{0}) = \gamma_1$	entropic E
2	F_-	-1	$+$	$v_2 = u^1 - \delta_+$	$\gamma_j = \begin{pmatrix} \rho\Delta \pm \\ \kappa p\Delta \pm \\ \epsilon\delta \pm \Delta \pm \\ -\epsilon\delta \pm \hat{H}^1\tilde{H} \\ \delta \pm \tilde{H} \end{pmatrix}$	magnetoacoustic
3	S_-	-1	$-$	$v_3 = u^1 - \delta_-$		
4	S_+	$+1$	$-$	$v_4 = u^1 + \delta_-$		fast $F_{\pm}$
5	F_+	$+1$	$+$	$v_5 = u^1 + \delta_+$		slow $S_{\pm}$
					$j = 2, 3, 4, 5.$	
6	A_-	-1		$v_6 = u^1 - \frac{H^1}{\sqrt{4\pi\rho}}$	$\gamma_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-H^3}{\sqrt{4\pi\rho}} \\ \frac{H^3}{\sqrt{4\pi\rho}} \\ -H^3 \\ H^3 \end{pmatrix}$	alfven $A_{\pm}$
7	A_+	$+1$		$v_7 = u^1 + \frac{H^1}{\sqrt{4\pi\rho}}$		
					$k = 6, 7$	

is convenient to write the vector $\mathbf{u}$ in the following form:

$$u = (\rho, p, u^1, \tilde{u}, \tilde{H}), \quad u = (u^1, \tilde{u}), \quad H = (H^1_0, \tilde{H}),$$

where by $\tilde{u}$ and $\tilde{H}$ we denote the two-dimensional vectors: $\tilde{u} = (u^2, u^3)$, $\tilde{H} = (H^2, H^3)$. Eigenvalues, eigenvectors and the corresponding seven types of disturbances permitted by the system of Eqs. (6.1) are set in Table I.[25,26]

The sign $\pm$ means here, that the wave goes in the right or in the left direction with respect to the medium. Taking into account simplifying the following calculations, we choose the particular form of nonhomogeneous simple element. Let covector λ^0 be a constant one form $\lambda^0 = dt$, then we have

$$\gamma_0 = B = \rho\omega(0,0,0,u^3,u^2,0,0).$$

We reduce here our considerations to the possibility of interaction of two magnetoacoustic simple waves F_- and F_+ in the nonhomogeneous system (6.1). We are interested here in solutions permitting existence of Riemann invariants. So we make an additional assumption, namely, we require vanishing of the commutator (5.2a) for the arbitrary $\gamma_0, \gamma_1, \gamma_2$ (i.e., for each $\alpha, \beta = 0,1,2, [\gamma_\alpha, \gamma_\beta] = 0$). The above demand constitutes the necessary and sufficient condition for the existence of a local curvilinear system of coordinates on the surface spanned by the fields $\gamma_0, \gamma_1, \gamma_2$ in the space $\mathscr{H}$. The curves are tangent to these fields, so for each $\alpha = 0.1,2$ we have (2.4), i.e.,

$$\begin{aligned}
&\frac{\partial\rho}{\partial R^0} = 0, \quad \frac{\partial\rho}{\partial R^1}\rho\Delta_-, && \frac{\partial\rho}{\partial R^2} = \rho\Delta_+, \\
&\frac{\partial p}{\partial R^0} = 0, \quad \frac{\partial p}{\partial R^1} = \kappa p\Delta_-, && \frac{\partial p}{\partial R^2} = \kappa p\Delta_+, \\
&\frac{\partial u^1}{\partial R^0} = 0, \quad \frac{\partial u^1}{\partial R^1} = \epsilon\delta_+\Delta_-, && \frac{\partial u^1}{\partial R^2} = \epsilon\delta_+\Delta_+, \\
&\frac{\partial u^2}{\partial R^0} = u^3, \quad \frac{\partial u^2}{\partial R^1} = \epsilon\delta_+\hat{H}^1H^2, && \frac{\partial u^2}{\partial R^2} = -\epsilon\delta_1\hat{H}^1H^2, \\
&\frac{\partial u^3}{\partial R^0} = u^2, \quad \frac{\partial u^3}{\partial R^1} = -\epsilon\delta_+\hat{H}^1H^3, && \frac{\partial u^3}{\partial R^2} = -\epsilon\delta_+\hat{H}_1H^3, \\
&\frac{\partial H^2}{\partial R^0} = 0, \quad \frac{\partial H^2}{\partial R^1} = \delta_+H^2, && \frac{\partial H^2}{\partial R^2} = \delta_+H^2, \\
&\frac{\partial H^3}{\partial R^0} = 0, \quad \frac{\partial H^3}{\partial R^1} = \delta_+H^3, && \frac{\partial H^3}{\partial R^2} = \delta_+H^3.
\end{aligned} \tag{6.2}$$

So the compatibility conditions (2.7a) in this particular case demand satisfying the relations

$$\lambda^0{}_{,R^\alpha} \wedge \lambda^\alpha = 0, \quad \alpha = 0,1,2,$$

which are automatically satisfied as λ^0 is a constant 1-form. Equations (6.2) allow us to separate two different situations. In the case when $\tilde{H} = 0$ the tangent surface is written in the parametric from as follows:

$$\rho = [(\kappa+1)/4\sqrt{A\kappa}](R^1 - R^2)^{2/(\kappa+1)}, \quad p = A\rho^\kappa,$$
$$\mathbf{H} = (H_0^1, 0, 0), \quad \mathbf{\Omega} = (\omega, 0, 0),$$
$$u^1 = \tfrac{1}{2}(R^1 - R^2), \quad u^2 = C_1 \cos\omega R^0 + C^2 \sin\omega R^0, \tag{6.3}$$
$$u^3 = D_1 \cos\omega R^0 + D_2 \sin\omega R^0,$$

where $A, \omega, H_0^1, C_i, D_i$, $i = 1,2$, are arbitrary constants. The wave covectors describing the local velocity and the direction of propagation have the form

$$\lambda^i = (\epsilon\delta - (R^1 + R^2)/2, -1), \quad \epsilon = \pm 1, i = 1,2,$$

where $\delta = v_i - u^1$ is the velocity of propagation of a wave relative to the medium which equals.

$$\delta = \sqrt{A\kappa}[(\kappa+1)/4\sqrt{A\kappa}(R^1 - R^2)^{(\kappa-1)/(\kappa-2)}].$$

The covector of the simple state equals $\lambda^0 = (1,0)$. The reduced system has the form

$$R^1{}_{,t} - [\delta - (R^1 + R^2)/2]R^1{}_{,x} = 0, \quad R^0 = t,$$
$$R^2{}_{,t} + [\delta + (R^1 + R^2)/2]R^2{}_{,x} = 0,$$

where r^1 and R^2 are the Riemann invariants for the interacting waves and R^0 is the parameter of the simple state. In this case the magnetic field $\mathbf{H}$ has the constant value and its direction is parallel to the vector $\mathbf{\Omega}$. The magnetoacoustic waves F_+ and F_- move in the same direction along $\mathbf{\Omega}$. The current $\mathbf{j}$ and the magnetic force $\mathbf{F}_m = \mathbf{j}\times\mathbf{H}$ vanish. It can be easily tested that equations of MHD reduce in this case to the hydrodynamics equations for rotating fluids.[26]

In the case $H_0^1 = 0$ the tangent surface is written in the parametric form

$$\int \left(A\kappa\rho^{\kappa-1} + \frac{C^2\rho}{4\pi}\right)^{1/2} \frac{d\rho}{\rho} = R^1 + R^2,$$
$$P = A\rho^\kappa, \quad \mathbf{H} = C\rho\mathbf{H}_0, \quad |\mathbf{H}_0| = 1, \tag{6.4}$$
$$u^1 = R^1 - R^2, \quad u^2 = C_1\cos\omega R^0 + C_2\sin\omega R^0,$$
$$U^3 = D_1\cos\omega R^0 + D_2\sin\omega R^0,$$
$$\mathbf{\Omega} = (\omega,0,0,), \quad \mathbf{H}_0\mathbf{\Omega} = 0,$$

where A, C, ω, C_i, D_i, $i = 1,2$, are arbitrary constants. The wave covectors of the interacting waves F_- and F_+ and the simple state are expressed as follows:

$$\lambda^i = (\epsilon\delta + R^2 - R^1, -1), \quad \lambda^0 = (1,0),$$
$$\epsilon = \pm 1, \quad i = 1,2.$$

where by $\delta = (A\kappa\rho^{\kappa-1} + C^2\rho/4\pi)^{1/2}$ we denote the velocity of propagation of the wave relative to the medium (see Table I). The reduced system of equations defining the equations in the Riemann invariants have the form

$$R^1{}_{,t} - (\delta + R^1 - R^2)R^1{}_{,x} = 0, \quad R^0 = t,$$
$$R^2{}_{,t} + (\delta - R^1 + R^2)R^2{}_{,x} = 0,$$

where R^1 and R^2 are Riemann invariants for the interacting waves and R^0 is the parameter of the simple state. From the form of the field velocity $\mathbf{u}$ it results that the trajectories of the fluid particles in the successive moments of time draw the spiral curve with the x axis parallel to the vector $\mathbf{\Omega}$. Curving of the particle trajectory is forced by the Coriolis force. The magnetic field $\mathbf{H}$ acts perpendicularly to the vector $\mathbf{\Omega}$. The quantity $|\mathbf{H}|/\rho$ is constant. From Eqs. (6.1c)–(6.1g) we can get the relation

$$\frac{d}{dt}\frac{\mathbf{H}}{\rho} = \left(\frac{\mathbf{H}}{\rho}\nabla\right)\mathbf{u},$$

which is satisfied identically. It means that the force lines of the magnetic field are inextensible, i.e., they remain "rigid." The current $\mathbf{j}$ flows in the direction perpendicular to the magnetic field $\mathbf{H}$. The magnetic force acts in the direction of the x axis, causing only the compression of the element of the fluid—it gives the contribution to the pressure [because $\nabla\mathbf{H}^2/8\pi \neq 0$, $(\mathbf{H}\nabla)\mathbf{H} = 0$]. In the physical sense the solution obtained describes the nonstationary nonrotational flow of the fluid on the spiral curve along the x axis. This type of flow can happen in the space between the conductive surfaces of two coaxial cylinders. If $\omega \to 0$ in Eq. (6.4), we obtain the solutions of the homogeneous equations (6.1) describing the interaction of two magnetoacoustic fast waves F_+ and F_-, which are written in the Riemann invariants form (found in Ref. 27).

In more general case the interaction of any magnetoacoustic simple wave with another one on the simple state (i.e., when in Eqs. (5.9) $C_{sr}^p \neq 0$) results in generating two new magnetoacoustic waves.[6,25,26] We can present it in a symbolic way

$$F_+ + F_- \to F'_+ + S'_+ + S'_- + F'_- .$$

As in this case the integrability conditions (5.9a) require that the Lie module containing any two fields, say γ_1 and γ_2 contain also other fields, say γ_3 and γ_4. It means the waves connected with fields γ_3 and γ_4 take part in the interaction. So as an effect of two simple waves interaction, we obtain new waves (of another type than those set in the initial moment). If these new waves do not vanish asymptotically for large t (i.e., $t \to \infty$), then the effect of generation is permanent (in accordance with accepted terminology, we are talking here about so-called nonelastic interaction[26]).

[1] A. Grundland and R. Zelazny, "Simple waves in quasilinear hyperbolic systems. I. Theory of simple waves and simple states. Examples of applications," J. Math. Phys. **24**, 2305 (1983).

[2] According to the summation convention.

[3] All the mappings and manifolds in this paper are assumed to be smooth enough to make our considerations sensible.

[4] A. Grundland, "Riemann invariants for nonhomogeneous systems of the first order partial quasilinear differential equations—algebraic aspects. Examples from gasdynamics," Arch. Mech. Stos. **26** (2), 274–296 (1974).

[5] A. Grundland, "Riemann invariants for nonhomogeneous systems of quasi-linear partial differential equations. Conditions of involution," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **22** (4), 177–185 (1974).

[6] A. Grundland and R. Żelazny, *Simple waves and Their Interactions in Quasilinear Hyperbolic Systems* (Polish Academy of Sciences, Publications of the Institute of Geophysics, P. W. N., Warsaw, 1982), pp. 1–109.

[7] The fact that vector X is proportional to Y will be denoted by $X \sim Y$.

[8] Let $C_\sigma(R) = (C^1(R), \ldots, C^n(R))$ be the set of vectors with the property $(C_\sigma(R), \lambda^\alpha(R)) = \delta_\sigma^\alpha$; then from Eqs. (2.6) we can determine the functions $\alpha_\sigma = (\partial\lambda^0/\partial R^\sigma, C_\sigma)$, $\beta'_\sigma = (\partial\lambda^r/\partial R^\sigma, C_r)$, $\zeta'_\sigma = (\partial\lambda^r/\partial R^\sigma, C_\sigma)$.

[9] F. John, *Partial Differential Equations* (Springer-Verlag, New York, 1975).

[10] Here we have used the notation $\lambda\cdot x \equiv \lambda_\mu x^\mu$ (the summation convention holds).

[11] If the system (2.20) is integrable, i.e., if Eqs. (2.11b) are satisfied then if $\mu = 1, \ldots, n$, where $n > k$, we can obtain a solution of Eqs. (2.11a) satisfying

the assumed condition at the point $R = R_0$, e.g., $\partial\lambda_\mu(R_0)/\partial R^s = \delta_{\mu s}$, $\lambda_\mu(R_0) = \delta_{\mu,k+1}$. Then the forms λ nd $\partial\lambda/\partial R^s$ are linearly independent in the neighborhood R_0.

[2]To simplify the formulae, we use the notation $Y := dY/dX$.

[3]Let E be a vector space and $E_1 := An\{\lambda^s\}$ and $E_2 := \langle \partial\xi(\tilde{\rho})/\partial\rho^t \rangle = T_x\Gamma$, be the subspace of E. If $V \in E_1 \cap E_2$, then $V = \Sigma_{t=1}^k \alpha^t \partial\xi/\partial\rho^t$ and $V\cdot\lambda^s = \Sigma_{t=1}^k \alpha^t(\lambda^s \partial\xi/\partial\rho^t) = 0$. So $E_1 \cap E_2 = \{0\}$ if $\Sigma_{t=1}^k \alpha^t(\lambda^s \partial\xi/\partial\rho^t) = 0$ possess only the zero solutions $\alpha^t = 0$, i.e., if $\det(\lambda^s \partial\xi/\partial\rho^t) \neq 0$. This is just the transversality condition.

[4]M. Burnat, "The hyperbolic double waves," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **17** (10) (1969).

[5]Z. Peradzyński, "Nonlinear plane k-waves and Riemann invariants," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **19** (9) (1971).

[6]F. John, "Formulation of singularities in one-dimensional non-linear wave propagation," Comm. Pure Appl. Math **27**, 377–405 (1974).

[7]B. Rozdestvenski and A. Sidorenko, "About impossibility of the gradient catastrophy for semilinear systems," J. Cal. Math. Math. Phys. **7** (5) (1967) (in Russian).

[8]We say that there is no gradient catastrophe if all partial derivatives $\partial R^j/\partial x^\mu$ of the solution u are bounded in the whole domain.

[9]A. Grundland, "Selected method of construction of solutions of systems of geophysics partial differential equations describing the wave propagation and interaction." Ph.D thesis Warsaw, 1978 (in Polish).

[20]A simple wave (3.2) (see Paper I) is of the ith type if $\gamma = df/dR$ is the ith eigenvector of matrix $a_j^\mu \lambda_\mu$.

[21]A. Grundland, "Algebraic properties of nonhomogeneous equations of magnetohydrodynamics in presence of gravitational and Coriolis forces. Examples of solutions—Simple states," Arch. Mech. Stos. **27** (1), 134–169 (1975).

[22]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley-Interscience, New York, 1962), Vol. 2.

[23]A. Jeffrey and T. Taniuti, *Nonlinear Wave Propagation* (Academic, New York, 1964).

[24]A. Jeffrey, *Quasilinear Hyperbolic Systems and Waves* (Pitman, London, 1976).

[25]H. Bretel and Z. Peradzyński, "Wave interaction and application to magnetohydrodynamics" (to be published).

[26]Z. Peradzyński, "Nonlinear interactions described by partial differential equations," Bull. Acad. Pol. Sci., Sér. Sci. Tech. **22** (5) (1974).

[27]W. Zajączkowski, "Some problems of double waves in magnetohydrodynamics," Arch. Mech. Stos. **26** (1974).

Generalized stochastic processes and continual observations in quantum mechanics

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We give here a mathematically rigorous form to an earlier work by Barchielli, Lanz, and Prosperi, in which it was found that a generalized stochastic process describes the results of continual observations of the position of a quantum particle. With the help of Albeverio and Høegh-Krohn's theory of Feynman path integrals, we define the characteristic functional of this process and demonstrate that it possesses the necessary properties of normalization, continuity, and positive definiteness. An explicit calculation of the Feynman path integral which defines the functional allows an analysis of the process to be made.

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I. INTRODUCTION

In a recent paper,[1] a formalism was developed for continual observations in quantum mechanics, and an expression for the joint probability distributions of the values of the observables which are continuously measured was given. The systems described in Ref. 1 have few degrees of freedom; the simplest model discussed is a particle whose position is continually observed. However, it was the authors' intention that the paper should provide a first step towards the solution of the problem of linking an objective description of a macrosystem to quantum mechanics. Such a problem is fundamental for a realistic interpretation of quantum mechanics, as is seen, for example, in the Daneri–Loinger–Prosperi theory of measurement[2] and in the axiomatic approach of Ludwig.[3] It seems necessary to give a mathematically precise form to the rather formal developments in Ref. 1, where Feynman path integrals are used without a too-rigorous concern for their definition. More specifically, we will show that (a) continual observations produce a generalized stochastic process, (b) the characteristic functional of the process can be rigorously defined by the use of Feynman path integrals, (c) the generalized stochastic process can be analyzed in terms of simpler processes, thus obtaining some physical insight.

Let us take a particle which, for simplicity of notation, moves on the line within the time interval $(0,t)$. The Hilbert space of the system is $L^2(R)$ and the Hamiltonian is $\hat{H} = \hat{p}^2/2m + \hat{V}$. The initial preparation of the system is described by the density matrix $\hat{W}_0$. The starting point of our work is the relation which, according to Ref. 1, gives the density of the joint probability distributions of the mean values of the position in ν time intervals into which the interval $(0,t)$ is subdivided (for simplicity we have taken intervals of equal size $\Delta t = t/\nu$):

$$p(\hat{W}_0|\bar{x}_1,(0,t_1);\bar{x}_2,(t_1,t_2);\cdots;\bar{x}_\nu,(t_{\nu-1},t)) = \mathrm{Tr}(\mathcal{F}(\bar{x}_\nu,\Delta t)\cdots\mathcal{F}(\bar{x}_1,\Delta t)\hat{W}_0), \tag{1.1}$$

where $\mathcal{F}(\bar{x},\Delta t)$ is an operator which acts on the space $\tau c(L^2(R))$ of the trace class operators on $L^2(R)$ as follows:

$$\mathcal{F}(\bar{x},\Delta t) = \lim_{n\to\infty}\left(\frac{\tilde{\alpha}\Delta t}{\pi n}\right)^{n/2}\int_{R^n} dx_1\cdots dx_n\,\delta\left(\bar{x}-\frac{\sum_{i=1}^n x_i}{n}\right) \times Q\left(x_n;\tilde{\alpha}\frac{\Delta t}{n}\right)U\left(\frac{\Delta t}{n}\right)\cdots \times Q\left(x_1;\frac{\tilde{\alpha}\Delta t}{n}\right)U\left(\frac{\Delta t}{n}\right), \tag{1.2}$$

with

$$Q\left(x;\frac{\tilde{\alpha}\Delta t}{n}\right)\hat{\rho} = \exp\left[-\frac{1}{2}\tilde{\alpha}\frac{\Delta t}{n}(\hat{q}-x)^2\right]\hat{\rho} \times\exp\left[-\frac{1}{2}\tilde{\alpha}\frac{\Delta t}{n}(\hat{q}-x)^2\right] \tag{1.3}$$

$$U\left(\frac{\Delta t}{n}\right)\hat{\rho} = \exp\left(-\frac{i}{\hbar}\frac{\Delta t}{n}\hat{H}\right)\hat{\rho} \times\exp\left(+\frac{i}{\hbar}\frac{\Delta t}{n}\hat{H}\right), \tag{1.4}$$

and $\tilde{\alpha}$ is a positive constant.

We give a brief illustration of relation (1.1); for a more complete explanation one is referred to the original paper. Making use of Gaussian instruments[4] for measuring the position of the particle, the authors are led to describe the results of K repeated observations, made at time $t_k = kt/K$, $k = 1, 2, \ldots, K$, with the following probability distribution:

$$P(\hat{W}_0|x(t_1)\in I_1,\ldots,x(t_K)\in I_K) = \mathrm{Tr}\left\{\left[\left(\frac{\alpha}{\pi}\right)^{K/2}\int_{I_1\times\cdots\times I_K} dx_1\cdots dx_K\,Q(x_K;\alpha) \times U\left(\frac{t}{K}\right)\cdots Q(x_1;\alpha)U\left(\frac{t}{K}\right)\right]\hat{W}_0\right\}, \tag{1.5}$$

where I_k, is any Borel set of R and α is a positive constant which specifies the degree of accuracy of the instruments. This relation recalls the corresponding prescription by Wigner, which deals with repeated observations of position with minimal disturbance resulting from the measuring apparatus:

$$P'(\hat{W}_0|x(t_1)\in I_1,\ldots,x(t_K)\in I_K) = \mathrm{Tr}\left[Q'(I_K)U\left(\frac{t}{K}\right)\cdots Q'(I_1)U\left(\frac{t}{K}\right)\hat{W}_0\right], \tag{1.6}$$

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where $Q'(I)\hat{\rho} = \hat{E}(I)\hat{\rho}\hat{E}(I)$ and where $\hat{E}(I)$ is the projection-valued measure of the operator $\hat{q}$. In contrast to the above, relation (1.5) defines a true probability distribution on R^K. However, it is still inadequate to construct a stochastic process describing the position of the particle during the time interval $(0,t)$; in fact, if observations are made at closer intervals, nonconsistent distributions are obtained. In order to overcome this difficulty, the authors suggest the following procedure. One considers the probability distributions for the time averages of the position in an arbitrary set of disjoint time intervals. Then one assumes $\alpha = \tilde{\alpha}t/K$, and takes the limit $K \to \infty$, i.e., one has in any finite time interval an infinite number of repeated observations each with a vanishing accuracy. This limit gives us relation (1.1).

We wish to define a generalized stochastic process[5] $\Phi(\varphi)$ over the nuclear space $\mathscr{D}(0,t)$[6] of test functions. As is known, such a process is thoroughly defined by its characteristic functional $L(\varphi)$, where $L(\varphi)$ is the mean of the random variable $\exp[i\Phi(\varphi)]$: $L(\varphi) = E(\exp[i\Phi(\varphi)])$. Such a functional must be continuous, normalized, and positive definite. To the mean value of the position during the time interval $(t_{\lambda-1}, t_\lambda)$ there corresponds the random variable $\Phi(\tilde{\chi}_{(t_{\lambda-1},t_\lambda)})$, where $\tilde{\chi}_{(t_{\lambda-1},t_\lambda)}(\tau)$ is the normalized characteristic function of the interval; strictly speaking, $\tilde{\chi}_{(t_{\lambda-1},t_\lambda)} \notin \mathscr{D}(0,t)$. From this it follows that the Fourier transform in R^ν of the joint probability distributions for the mean values within ν intervals, gives the following formal expression for $L(\varphi)$ when $\varphi(\tau) = \Delta t \sum_{\lambda=1}^{\nu} \varphi_\lambda \tilde{\chi}_{(t_{\lambda-1},t_\lambda)}(\tau)$, where $\Delta t = t/\nu$:

$$L\left(\Delta t \sum_1^\nu \varphi_\lambda \tilde{\chi}_{(t_{\lambda-1},t_\lambda)}\right) = \int_{R^\nu} d\bar{x}_1 .. d\bar{x}_\nu \, \mathrm{Tr}(\mathscr{F}(\bar{x}_\nu,\Delta t)\cdots\mathscr{F}(\bar{x}_1,\Delta t)\hat{W}_0) \times \exp\left(i\Delta t \sum_1^\nu \varphi_\lambda \bar{x}_\lambda\right). \tag{1.7}$$

One could investigate the existence of the limit in (1.2) using operatorial techniques (the Lie–Kato–Trotter product formula), and define $L(\varphi)$ by extension for every $\varphi \in \mathscr{D}(0,t)$. However, we prefer to take another path, which indicates the possibilities of the use of Feynman path integrals in problems of this type. Relation (1.7) suggests fairly directly a formal expression of $L(\varphi)$ by using Feynman path integrals. We shall demonstrate that a precise meaning can be given to such an expression within the framework of Albeverio and Høegh-Krohn's theory of Fresnel integrals[7] and that the functional thus defined has the necessary properties: This will be discussed in Secs. 2 and 3. In Sec. 4, an explicit calculation of the Fresnel integral giving $L(\varphi)$ allows us to analyze the process in terms of simpler processes. It should be noted, however, that, in this decomposition, alongside with some true stochastic processes—a white noise and a Brownian motion for the velocity of the particle caused by the observations of the position—there appear "processes" with "probability distributions" which can be negative. In the free case, deterministic processes appear concentrated on the trajectories of a classical free particle, with initial "distribution" in phase space given by the Wigner distribution associated with $\hat{W}_0$. Finally, the presence of a potential introduces a "jump process" for the velocity of the particle. Besides, let us observe that the hypothesis we shall be making on the potential V excludes the case of the harmonic oscillator. Some observations on this subject are to be found at the end of the paper.

2. DEFINITION OF THE CHARACTERISTIC FUNCTIONAL OF THE PROCESS

To guess the structure of $L(\varphi)$, we start from relation (1.7) which we rewrite introducing the Fourier transform of $\mathscr{F}(\bar{x},\Delta t)$. Note that one can write

$$\mathscr{F}(\bar{x},\Delta t) = \frac{\Delta t}{2\pi}\int_{-\infty}^{+\infty} dK \exp(-iK\,\Delta t\,\bar{x})\exp(\mathscr{K}_K \Delta t), \tag{2.1}$$

where

$$\mathscr{K}_K\hat{\rho} = -(i/\hbar)[\hat{H},\hat{\rho}] - \tfrac{1}{4}\tilde{\alpha}[\hat{q},[\hat{q},\hat{\rho}]] + \tfrac{1}{2}iK\{\hat{q},\hat{\rho}\} - (K^2/4\tilde{\alpha})\hat{\rho} \tag{2.2}$$

and $\{\hat{A},\hat{B}\}$ denotes the anticommutator between $\hat{A}$ and $\hat{B}$; a demonstration due to Frigerio is indicated in Appendix A. Expression (1.7) can then be rewritten as follows:

$$L\left(\Delta t \sum_{\lambda=1}^{\nu} \varphi_\lambda \tilde{\chi}_{(t_{\lambda-1},t_\lambda)}\right) = \mathrm{Tr}[\exp(\mathscr{K}_{\varphi_\nu}\Delta t)\cdots\exp(\mathscr{K}_{\varphi_1}\Delta t)\hat{W}_0]. \tag{2.3}$$

For an arbitrary test function $\varphi \in \mathscr{D}(0,t)$, the following definition follows naturally:

$$L(\varphi) = \mathrm{Tr}\left(\left\{\lim_{n\to\infty}\left[\exp\left(\mathscr{K}_{\varphi_n}\frac{t}{n}\right)\cdots\exp\left(\mathscr{K}_{\varphi_j}\frac{t}{n}\right)\times\cdots\exp\left(\mathscr{K}_{\varphi_1}\frac{t}{n}\right)\right]\right\}\hat{W}_0\right), \tag{2.4}$$

where $\varphi_j = \varphi(jt/n)$ and where the chronological ordering in the product of operators is essential. With standard calculations, representing the generalized eigenfunctions of $\hat{q}$ with $|q\rangle$, we obtain

$$L(\varphi) = \lim_{n\to\infty}\int dq_0\, dq_0' \cdots dq_n\, dq_n'\, \delta(q_n - q_n')\langle q_0|\hat{W}_0|q_0'\rangle \times\prod_{j=1}^{n}\left(\exp\left\{-\frac{t}{n}\left[\frac{\tilde{\alpha}}{4}(q_j-q_j')^2 - \frac{i}{2}\varphi_j(q_j+q_j') + \frac{\varphi_j^2}{4\tilde{\alpha}}\right]\right\} \times\exp\left\{\frac{i}{\hbar}\frac{t}{n}\sum_{j=1}^{n}\left[\frac{m}{2}\left(\frac{q_j-q_{j-1}}{t/n}\right)^2 - V\left(\frac{q_j+q_{j-1}}{2}\right)\right]\right\} \times\exp\left\{-\frac{i}{\hbar}\frac{t}{n}\sum_{j=1}^{n}\left[\frac{m}{2}\left(\frac{q_j'-q_{j-1}'}{t/n}\right)^2 - V\left(\frac{q_j'+q_{j-1}'}{2}\right)\right]\right\}\right)$$

$$= \exp\left[-\frac{1}{4\tilde{\alpha}} \int \varphi^2(\tau)\, d\tau \right] \int_R dq \int_{q(t)=q'(t)=q} \mathscr{D}[q(\tau)]\mathscr{D}[q'(\tau)]$$
$$\times \exp\left\{ \frac{im}{2\hbar} \int_0^t d\tau\, [\dot{q}^2(\tau) - \dot{q}'^2(\tau)] \right\} \langle q(0)|\hat{W}_0|q'(0)\rangle$$
$$\times \exp\left\{ -\frac{i}{\hbar} \int d\tau\, [V(q(\tau)) - V(q'(\tau))] \right\} \exp\left[-\frac{\tilde{\alpha}}{4} \int_0^t [q(\tau) - q'(\tau)]^2\, d\tau \right]$$
$$\times \exp\left[i \int_0^t \varphi(\tau) \frac{q(\tau) + q'(\tau)}{2}\, d\tau \right]. \tag{2.5}$$

From this point onwards, we shall concern ourselves with a mathematically rigorous treatment. We shall give meaning to the formal Feynman integral in (2.5) using a Fresnel integral on a real Hilbert space $\mathscr{H}$.[7] We begin with giving some definitions and recalling a few basic facts about the Fresnel integral. Let $\mathfrak{H}$ be the real Hilbert space of continuous functions $\gamma(\tau)$ $[\tau \in (0,t)]$ with weak derivative $d\gamma/d\tau \equiv \dot{\gamma} \in L^2(0,t)$ and $\gamma(t) = 0$, with inner product $(\gamma_1,\gamma_2)_{\mathfrak{H}} = (m/\hbar)\int_0^t \dot{\gamma}_1(\tau)\dot{\gamma}_2(\tau)\, d\tau$. Let $\mathscr{H} = \mathfrak{H}^+ \oplus \mathfrak{H}^-$ $(\mathfrak{H}^+ = \mathfrak{H}^- = \mathfrak{H})$. An element of $\mathscr{H}$ will be denoted $\Gamma \equiv [\gamma^+,\gamma^-]$; we shall denote by $(\cdot,\cdot)$ the inner product in $\mathscr{H}$. Let $B = 1_{\mathfrak{H}^+} \oplus (-1_{\mathfrak{H}^-})$: observe that $B = B^* = B^{-1}$. The vectors $\gamma_\sigma \in \mathfrak{H}$ are defined by $\gamma_\sigma(\tau) = t - \sigma \vee \tau$ with $\sigma \vee \tau = \max(\sigma,\tau)$. The following relations hold: $(\hbar/m)(\gamma,\gamma_\sigma)_{\mathfrak{H}} = \gamma(\sigma)$ and $(\hbar/m)(\gamma_\sigma,\gamma_\tau) = t - \sigma \vee \tau$ $[(\hbar/m)(t - \sigma \vee \tau)$ is the reproducing kernel of $\mathfrak{H}]$. Further let $\Gamma_\sigma^+ = [\gamma_\sigma,0] \in \mathscr{H}$, $\Gamma_\sigma^- = [0,\gamma_\sigma] \in \mathscr{H}$; for $\Gamma_\sigma^{(\pm)}$ analogous relations hold true. With the notations of the Fresnel integral via Ref. 7, we are now led to the following definition:

$$L(\varphi) = \exp\left[-\frac{1}{4\tilde{\alpha}} \int_0^t \varphi^2(\tau)\, d\tau \right] \cdot \int_R dq \left\{ \exp\left[iq \int_0^t \varphi(\tau)\, d\tau \right] \cdot \widetilde{\int_{\mathscr{H}}} d\Gamma \left(\exp\left[\frac{i}{2} (\Gamma,B\Gamma) \right] \cdot \langle \gamma^+(0) + q|\hat{W}_0|\gamma^-(0) + q\rangle \right.\right.$$
$$\times \exp\left[-\frac{i}{\hbar} \int_0^t [V(\gamma^+(\tau) + q) - V(\gamma^-(\tau) + q)]\, d\tau \right]$$
$$\left.\left. \times \exp\left[-\frac{\tilde{\alpha}}{4} \int_0^t [\gamma^+(\tau) - \gamma^-(\tau)]^2\, d\tau \right] \exp\left[i \int_0^t \varphi(\tau) \frac{\gamma^+(\tau) + \gamma^-(\tau)}{2}\, d\tau \right] \right) \right\}. \tag{2.6}$$

The Fresnel integral $\int_{\mathscr{H}} d\Gamma \exp[(i/2)(\Gamma,B\Gamma)] f(\Gamma)$ is defined by [see pp. 59, 60 of Ref. 7]

$$\widetilde{\int_{\mathscr{H}}} d\Gamma \exp\left[\frac{i}{2} (\Gamma,B\Gamma) \right] f(\Gamma) \overset{\text{def}}{=} \int_{\mathscr{H}} \exp\left[-\frac{i}{2} (\Gamma',B\Gamma') \right] d\mu_f(\Gamma'), \tag{2.7}$$

for every f such that

$$f(\Gamma) = \int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')]\, d\mu_f(\Gamma'), \tag{2.8}$$

where μ_f is a complex measure on $(\mathscr{H},\mathscr{B})$ with bounded total variation; we have denoted with $\mathscr{B}$ the σ-algebra associated with the metric of $\mathscr{H}$. Let $\mathscr{M}(\mathscr{H})$ be the Banach space of such measures with the total variation norm. $\mathscr{M}(\mathscr{H})$ is in fact a commutative Banach algebra under the convolution product of the measures. The mapping $\mathscr{J}: \mu_f \to f$ is linear and one-to-one. Let $\mathscr{F}(\mathscr{H})$ be the space of functions of type (2.8). With the norm $\|f\|_0 = \|\mu_f\|$ and under usual multiplication it is a Banach algebra; $\mathscr{J}$ is an isomorphism between Banach algebras. The functional $I(f) = \widetilde{\int}_{\mathscr{H}} \exp[(i/2)(\Gamma,B\Gamma)] f(\Gamma)\, d\Gamma$ on the Banach algebra $\mathscr{F}(\mathscr{H})$ is continuous.

In order to guarantee that (2.6) is a good definition of $L(\varphi)$ two hypotheses are required. First, we shall require that the potential V be the Fourier transform of a bounded complex measure:

$$V(x) = \int_R \exp(i\alpha x)\, d\mu(\alpha). \tag{2.9}$$

It is seen that a function of type (2.9) is certainly bounded, and therefore such a hypothesis is physically restrictive because, for example, it excludes the case of a harmonic oscillator. Secondly, choosing for simplicity $\hat{W}_0 = |\psi_0\rangle\langle\psi_0|$ we shall require

$$\psi_0(x) = \int_R \exp(i\alpha x)\, d\nu(\alpha) \tag{2.10}$$

with ν a bounded complex measure. Observe that the linear combinations of such pure states are dense in the state space.

Proposition 2.1: Let $\mathscr{H}$ and B be defined as above. Let $V(x)$ and $\psi_0(x)$ be Fourier transforms of bounded complex measures on R. Then $L(\varphi)$ is well defined by (2.6).

Proof: Let us demonstrate that the factors appearing in the Fresnel integral in (2.6) belong to the Banach algebra $\mathscr{F}(\mathscr{H})$. This shows that the Fresnel integral exists:

$$\text{(I)} \quad \langle \gamma^+(0) + q|\psi_0\rangle\langle\psi_0|\gamma^-(0) + q\rangle = \int_{R^2} d\nu(\alpha)\, d\nu^*(\alpha') \exp[i(\alpha - \alpha')q] \exp\left[i\alpha \frac{\hbar}{m} (\Gamma,\Gamma_0^+) \right] \exp\left[-i\alpha' \frac{\hbar}{m} (\Gamma,\Gamma_0^-) \right]$$
$$= \int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')]\, d\chi(\Gamma'), \tag{2.11}$$

where

$$\chi(\Gamma') = \int_{R^2} d\nu(\alpha)\, d\nu^*(\alpha') \exp[i(\alpha - \alpha')q]\delta\left(\Gamma' - \alpha\frac{\hbar}{m}\Gamma_0^+ - \alpha'\frac{\hbar}{m}\Gamma_0^-\right). \tag{2.12}$$

(II) $$\int_0^t [V(\gamma^+(\tau)+q) - V(\gamma^-(\tau)+q)]\, d\tau = \int_0^t d\tau \int_R d\mu(\beta) \exp(iq\beta)\left\{\exp\left[i\beta\frac{\hbar}{m}(\Gamma,\Gamma_\tau^+)\right] - \exp\left[i\beta\frac{\hbar}{m}(\Gamma,\Gamma_\tau^-)\right]\right\}$$
$$= \int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')]\, d\xi(\Gamma'), \tag{2.13}$$

where

$$\zeta(\Gamma') = \int_0^t d\tau \int_R d\mu(\beta) \exp(iq\beta)\cdot\left[\delta\left(\Gamma' - \beta\frac{\hbar}{m}\Gamma_\tau^+\right) - \delta\left(\Gamma' + \beta\frac{\hbar}{m}\Gamma_\tau^-\right)\right], \tag{2.14}$$

and hence

$$\exp\left\{-\frac{i}{\hbar}\int_0^t d\tau[V(\gamma^+(\tau)+q) - V(\gamma^-(\tau)+q)]\right\} = \int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')]\, d\exp\left[-\frac{i}{\hbar}\zeta\right](\Gamma'), \tag{2.15}$$

where $\exp[-(i/\hbar)\zeta]$ is defined by the covergent series in $\mathscr{M}(\mathscr{H})$:

$$\exp[-(i/\hbar)\zeta](\Gamma') = \delta(\Gamma') - (i/\hbar)\zeta(\Gamma') + \tfrac{1}{2}(-i/\hbar)^2(\zeta * \zeta)(\Gamma') + \cdots. \tag{2.16}$$

(III) Let us consider the third factor; the integral in the exponent, $h(\Gamma) = \int_0^t [\gamma^+(\tau) - \gamma^-(\tau)]^2\, d\tau$, is the quadratic form of a positive semidefinite Hermitian operator which we denote by S, $(\Gamma, S\Gamma) = \int_0^t [\gamma^+(\tau) - \gamma^-(\tau)]^2\, d\tau$ {in particular, it is easy to check that $h(\Gamma+\Gamma') + h(\Gamma-\Gamma') = 2[h(\Gamma) + h(\Gamma')]$}. If, moreover, S belongs to $\tau c(\mathscr{H})$, on the basis of a general result,[8] $\exp[-\frac{1}{4}\tilde{\alpha}(\Gamma, S\Gamma)]$ is the Fourier transform of a Gaussian probability (with mean 0) on $\mathscr{H}$. Let us ascertain the validity of this condition. The set of vectors $\{e_n, n = 0, 1, \cdots\}$, where

$$e_0 = \left(\frac{\hbar}{mt}\right)^{1/2}(t-\tau), \quad e_n = \left(\frac{2\hbar t}{m}\right)^{1/2}\frac{1}{n\pi}\sin\left(\frac{n\pi}{t}\tau\right),$$

is a basis in $\mathfrak{H}$. The set $\{[e_n,0], [0,e_n]\} = \{E_n^+, E_n^-\}$ is a basis in $\mathscr{H}$. Since

$$\sum_{n=0}^{\infty}[(SE_n^+, E_n^+) + (SE_n^-, E_n^-)] = 2\sum_{n=0}^{\infty}\int_0^t e_n^2(\tau)d\tau = 2\left[\frac{\hbar}{m}\frac{t^3}{3} + \sum_{n=1}^{\infty}\frac{\hbar t^2}{m\pi^2}\frac{1}{n^2}\right] < +\infty,$$

S is a trace-class operator. We can therefore write

$$\exp\left[-\frac{\tilde{\alpha}}{4}\int_0^t [\gamma^+(\tau) - \gamma^-(\tau)]^2\, d\tau\right] = \int_{\mathscr{H}} \exp[i(\Gamma,\Gamma')]\, d\tilde{\rho}(\Gamma'), \tag{2.17}$$

where $\tilde{\rho}$ is a Gaussian probability on $\mathscr{H}$. Let ρ be the measure on $\mathscr{H}$ such that, for every $\tilde{\rho}$-integrable function g, the following relation holds:

$$\int_{\mathscr{H}} g(\Gamma')\, d\tilde{\rho}(\Gamma') = \int_{\mathscr{H}} g(B\Gamma')\, d\rho(\Gamma'). \tag{2.18}$$

In conclusion we obtain

$$\exp\left[-\frac{\tilde{\alpha}}{4}\int_0^t [\gamma^+(\tau) - \gamma^-(\tau)]^2\, d\tau\right] = \int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')]\, d\rho(\Gamma'), \tag{2.19}$$

where ρ is a Gaussian probability. Observe the following property of ρ relevant for the ensuing arguments. Let $\mathscr{H}_s$ be the subspace of $\mathscr{H}$ defined by $\mathscr{H}_s = \{[\gamma^+,\gamma^-]; \gamma^+ = \gamma^-\}$. From $\int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')\, d\rho(\Gamma') = 1$ for every $\Gamma \in \mathscr{H}_s$, and from the fact that ρ is a probability, we can deduce that it is concentrated on $\mathscr{H}_s$.

(IV) $$\exp\left[i\int_0^t \frac{\varphi(\tau)}{2}[\gamma^+(\tau) + \gamma^-(\tau)]\, d\tau = \exp\left[i\frac{\hbar}{m}\int_0^t \frac{\varphi(\tau)}{2}[(\Gamma,\Gamma_\tau^+) + (\Gamma,\Gamma_\tau^-)]\, d\tau\right.\right.$$
$$= \int_{\mathscr{H}} \exp[i(\Gamma,B\Gamma')]\, d\psi(\Gamma'), \tag{2.20}$$

where

$$\psi(\Gamma') = \delta\left(\Gamma' - \frac{\hbar}{m}\int_0^t \frac{\varphi(\tau)}{2}(\Gamma_\tau^+ - \Gamma_\tau^-)d\tau\right) \tag{2.21}$$

Since $\Gamma_\tau^{(\pm)}$ is continuous in τ in the norm of $\mathscr{H}$, the definition of the integral in (2.21) gives no problems. We conclude that the Fresnel integral in (2.6) is well defined by

$$\int_{\mathscr{H}} \exp\left[\frac{i}{2}(\Gamma,B\Gamma)\right]\langle\gamma^+(0)+q|\psi_0\rangle\langle\psi_0|\gamma^-(0)+q\rangle\exp\left\{-\frac{i}{\hbar}\int_0^t d\tau[V(\gamma^+(\tau)+q)-V(\gamma^-(\tau)+q)]\right\}$$
$$\times\exp\left[-\frac{\tilde{\alpha}}{4}\int_0^t[\gamma^+(\tau)-\gamma^-(\tau)]^2\,d\tau\right]\cdot\exp\left\{i\int_0^t\frac{\varphi(\tau)}{2}[\gamma^+(\tau)+\gamma^-(\tau)]\,d\tau\right\}d\Gamma$$
$$\overset{\text{def.}}{=}\int_{\mathscr{H}}\exp\left[-\frac{i}{2}(\Gamma',B\Gamma')\right]d\left(\chi*\exp\left(-\frac{i}{\hbar}\zeta\right)*\rho*\psi\right)(\Gamma'), \tag{2.22}$$

where, χ,ζ,ρ, and ψ are defined by (2.12), (2.14), (2.19), and (2.21). It remains to discuss the existence of the integral in q. Since this follows directly from the arguments in Sec. 3, we can here take this point for granted. ■

Let us close this section by giving an equivalent expression of the Fresnel integral in (2.22), linked to the decomposition $\mathscr{H}=\mathscr{H}_s\oplus\mathscr{H}_s^\perp$. Let $\Gamma=[\gamma^+,\gamma^-]\in\mathscr{H}$. Then $\Gamma=\Sigma+A$, where

$$\Sigma=\left[\frac{\gamma^++\gamma^-}{2},\frac{\gamma^++\gamma^-}{2}\right]\in\mathscr{H}_s,\quad A=\left[\frac{\gamma^+-\gamma^-}{2},-\frac{\gamma^+-\gamma^-}{2}\right]\in\mathscr{H}_s^\perp.$$

Let us define $\Sigma_\sigma=[\gamma_\sigma,\gamma_\sigma]$ and $A_\sigma=[\gamma_\sigma,-\gamma_\sigma]$. Note that $B\mathscr{H}_s=\mathscr{H}_s^\perp$, $BA_\sigma=\Sigma_\sigma$. One can easily verify that the integral on $\mathscr{H}$ in (2.22) may be expressed as an integral on the product space $\mathscr{H}_s\times\mathscr{H}_s^\perp$, metrically equivalent to $\mathscr{H}$, as follows:

$$\int_{\mathscr{H}}\exp\left[-\frac{i}{2}(\Gamma',B\Gamma')\right]d\left(\chi*\exp\left[-\frac{i}{\hbar}\zeta\right]*\rho*\psi\right)(\Gamma')=\int_{\mathscr{H}_s\times\mathscr{H}_s^\perp}\exp[-i(\Sigma',BA')]d\left(\epsilon*\exp\left[-\frac{i}{\hbar}\eta\right]*\omega*\lambda\right)(\Sigma',A'), \tag{2.23}$$

where

$$\epsilon(\Sigma',A')=\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')\exp[iq(\alpha-a')]\delta\left(\Sigma'-\frac{\hbar}{m}\frac{\alpha+\alpha'}{2}\Sigma_0\right)\times\delta\left(A'-\frac{\hbar}{m}\frac{\alpha-\alpha'}{2}A_0\right), \tag{2.24}$$

$$\eta(\Sigma',A')=\int_0^t d\tau\int_R d\mu(\beta)\exp(iq\beta)\delta\left(A'-\frac{\hbar}{m}\frac{\beta}{2}A_\tau\right)\times\left[\delta\left(\Sigma'-\frac{\hbar}{m}\frac{\beta}{2}\Sigma_\tau\right)-\delta\left(\Sigma'+\frac{\hbar}{m}\frac{\beta}{2}\Sigma_\tau\right)\right], \tag{2.25}$$

$$\lambda(\Sigma',A')=\delta(\Sigma')\times\delta\left(A'-\frac{\hbar}{m}\int_0^t\varphi(\tau)\frac{A_\tau}{2}d\tau\right), \tag{2.26}$$

$$\omega(\Sigma',A')=\omega_s(\Sigma')\times\delta(A'); \tag{2.27}$$

in relation (2.27), ω_s is a probability on $\mathscr{H}_s$ such that

$$\exp\left[-\frac{\tilde{\alpha}}{4}\int_0^t[\gamma^+(\tau)-\gamma^-(\tau)]^2\,d\tau\right]=\int_{\mathscr{H}_s}\exp[i(\Sigma',BA)\,d\omega_s(\Sigma'), \tag{2.28}$$

where

$$A=\left[\frac{\gamma^+-\gamma^-}{2},-\frac{\gamma^+-\gamma^-}{2}\right].$$

3. EXISTENCE OF A GENERALIZED STOCHASTIC PROCESS WITH $L(\varphi)$ AS CHARACTERISTIC FUNCTIONAL

In this section we will show that the functional $L(\varphi)$ over the nuclear space $\mathscr{D}(0,t)$, defined in Sec. 2, has the following properties:

(1) $L(0)=1$ (normalization),

(2) $L(\varphi)$ is continuous

(3) for any functions $\varphi^{(1)},\ldots\varphi^{(n)}$ in $\mathscr{D}(0,t)$ and any complex numbers $\alpha_1,\ldots\alpha_n$, one has $\Sigma_{j,k=1}^n L(\varphi^{(j)}-\varphi^{(k)})\alpha_j\alpha_k^*\geqslant 0$ (positive definiteness).

These properties guarantee the existence of a generalized process over the nuclear space of test functions $\mathscr{D}(0,t)$.[5]

The proof of the normalization property requires the explicit calculation of $L(\varphi)$ for $\varphi=0$. To avoid later repetitions, we here give the calculation for an arbitrary $\varphi\in\mathscr{D}(0,t)$. We have

$$L(\varphi)=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_R dq\exp\left[iq\int_0^t\varphi(\tau)\,d\tau\right]\int_{\mathscr{H}_s\times\mathscr{H}_s^\perp}\exp[-i(\Sigma,BA)]d\left(\epsilon*\omega*\lambda*\exp\left[-\frac{i}{\hbar}\eta\right]\right)(\Sigma,A)$$
$$=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_R dq\exp\left[iq\int_0^t\varphi(\tau)\,d\tau\right]\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')\sum_{n=0}^\infty\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\int_{R^n}\prod_{i=1}^n d\tau_i\int_{R^n}\prod_{i=1}^n d\mu(\beta_i)$$
$$\times\exp\left[iq\left(\alpha-\alpha'+\sum_{i=1}^n\beta_i\right)\right]\cdot\int_{\mathscr{H}_s^{n+3}\times\mathscr{H}_s^{n+3}}\exp\left[-i\left(\Sigma+\tilde{\Sigma}+\overline{\Sigma}+\sum_{i=1}^n\Sigma_i,B\left(A+\tilde{A}+\overline{A}+\sum_{i=1}^n A_i\right)\right)\right]$$
$$\times d\left(\delta\left(\Sigma-\frac{\hbar}{m}\frac{\alpha+\alpha'}{2}\Sigma_0\right)\times\delta\left(A-\frac{\hbar}{m}\frac{\alpha-\alpha'}{2}A_0\right)\times\omega_s(\tilde{\Sigma})\times\delta(\tilde{A})\times\delta(\overline{\Sigma})\times\delta\left(\overline{A}-\frac{\hbar}{m}\int_0^t\frac{\varphi(\tau)}{2}A_\tau\,d\tau\right)\right.$$

$$\times\prod_{i=1}^{n}\left\{\left[\delta\left(\Sigma_i-\frac{\hbar}{m}\frac{\beta_i}{2}\Sigma_{\tau_i}\right)-\delta\left(\Sigma_i+\frac{\hbar}{m}\frac{\beta_i}{2}\Sigma_{\tau_i}\right)\right]\times\delta\left(A_i-\frac{\hbar}{m}\frac{\beta_i}{2}A_{\tau_i}\right)\right\}\Bigg)$$

$$=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_R dq\exp\left[iq\int_0^t\varphi(\tau)\,d\tau\right]\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\int_{R^n}\prod_{i=1}^{n}d\tau_i\int_{R^n}\prod_{i=1}^{n}d\mu(\beta_i)$$

$$\times\exp\left[iq\left(\alpha-\alpha'+\sum_{i=1}^{n}\beta_i\right)\right]\cdot\exp\left\{-i\frac{\hbar}{m}\frac{\alpha+\alpha'}{2}\left[(\alpha-\alpha')t+\sum_{i=1}^{n}\beta_i(t-\tau_i)+\int_0^t\varphi(\tau)(t-\tau)\,d\tau\right]\right\}$$

$$\times\exp\left\{-\frac{\tilde{\alpha}}{4}\int_0^t d\tau\left[\frac{\hbar}{m}(\alpha-\alpha')(t-\tau)+\frac{\hbar}{m}\sum_{i=1}^{n}\beta_i(t-\tau\vee\tau_i)+\frac{\hbar}{m}\int_0^t\varphi(\sigma)(t-\tau\vee\sigma)\,d\sigma\right]^2\right\}$$

$$\times\prod_{i=1}^{n}\left(\exp\left\{-\frac{i\hbar}{m}\frac{\beta_i}{2}\left[(\alpha-\alpha')(t-\tau_i)+\sum_{l=1}^{n}\beta_l(t-\tau_i\vee\tau_l)+\int_0^t\varphi(\tau)(t-\tau_i\vee\tau)\,d\tau\right]\right\}-\text{c.c.}\right). \tag{3.1}$$

In the first step it was taken into account that, for any bounded continuous functional $f(\Gamma)$, $\int_{\mathscr{H}}f(\Gamma)\,d(\mu*\nu)(\Gamma)=\int_{\mathscr{H}\times\mathscr{H}}f(\Gamma+\Gamma')\,d\mu(\Gamma)\,d\nu(\Gamma')$ holds. The exchange of the series with the integral on $\mathscr{H}$ is possible since $\int_{\mathscr{H}}\exp[-\frac{1}{2}i(\Gamma,B\Gamma)]\,d\mu(\Gamma)$ is a continuous functional on the Banach algebra $\mathscr{M}(\mathscr{H})$.

Proposition 3.1: $L(\varphi)$ is normalized, i.e., $L(0)=1$.

Proof: Take $\varphi=0$ in expression (3.1). The first term of the series, $n=0$, may be calculated as follows:

$$\lim_{L\to\infty}\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')\int_{-L}^{+L}dq\exp[iq(\alpha-\alpha')]$$

$$\times\exp\left[-i\frac{\hbar}{m}\frac{\alpha+\alpha'}{2}(\alpha-\alpha')t\right]$$

$$\times\exp\left[-\frac{\tilde{\alpha}}{4}\left(\frac{\hbar}{m}\right)^2(\alpha-\alpha')^2\int_0^t(t-\tau)^2\,d\tau\right]=1, \tag{3.2}$$

where we have taken into account that

$$1=\lim_{L\to\infty}\int_{-L}^{+L}\psi_0(q)\psi_0^*(q)\,dq$$

$$=\lim_{L\to\infty}\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')\int_{-L}^{L}\exp[i(\alpha-\alpha')q]\,dq$$

$$=\lim_{L\to\infty}\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')\,2\pi\delta_L(\alpha-\alpha').$$

We shall refrain here (and in what follows) from a detailed discussion of the use of the delta function, since it would become somewhat lengthy.

Let us calculate now the general term relative to $n\neq 0$:

$$\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\lim_{L\to\infty}\int_{R^2}d\nu(\alpha)\,d\nu^*(\alpha')$$

$$\times\int_{R^n}\prod_{i=1}^{n}d\tau_i\int_{R^n}\prod_{i=1}^{n}d\mu(\beta_i)\,2\pi\delta_L\left(\alpha-\alpha'+\sum_{i=1}^{n}\beta_i\right)$$

$$\times\exp\left\{-i\frac{\hbar}{m}\frac{\alpha+\alpha'}{2}\left[(\alpha-\alpha')t+\sum_{i=1}^{n}\beta_i(t-\tau_i)\right]\right\}$$

$$\times\exp\left\{-\frac{\tilde{\alpha}}{4}\int_0^t\left[\frac{\hbar}{m}(\alpha-\alpha')(t-\tau)\right.\right.$$

$$\left.\left.+\frac{\hbar}{m}\sum_{i=1}^{n}\beta_i(t-\tau\vee\tau_i)\right]^2d\tau\right\}$$

$$\times\prod_{i=1}^{n}\left(\exp\left\{-i\frac{\hbar}{m}\frac{\beta_i}{2}\left[(\alpha-\alpha')(t-\tau_i)\right.\right.\right.$$

$$\left.\left.\left.+\sum_{l=1}^{n}\beta_l(t-\tau_i\vee\tau_l)\right]\right\}-\text{c.c.}\right)=0. \tag{3.3}$$

In fact, under the condition $\alpha-\alpha'+\Sigma_{i=1}^{n}\beta_i=0$ resulting from the presence of $\delta(\alpha-\alpha'+\Sigma_{i=1}^{n}\beta_i)$, one factor in the last product is always zero: Observe that, since then

$$(\alpha-\alpha')(t-\tau_i)+\sum_{l=1}^{n}\beta_l(t-\tau_i\vee\tau_l)$$

$$=\sum_{\tau_l>\tau_i}\beta_l[(t-\tau_l)-(t-\tau_i)],$$

if i is such that $\tau_i\geqslant\tau_l$, $\forall l$, one has

$$\exp\left\{-i\frac{\hbar}{m}\frac{\beta_i}{2}\left[(\alpha-\alpha')(t-\tau_i)\right.\right.$$

$$\left.\left.+\sum_{l=1}^{n}\beta_l(t-\tau_i\vee\tau_l)\right]\right\}-\text{c.c.}=0.\blacksquare$$

Proposition 3.2: $L(\varphi)$ is continuous.

Proof:

$$L(\varphi)=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]$$

$$\times\int_R\exp\left[iq\int_0^t\varphi(\tau)\,d\tau\right]F_q(\varphi)\,dq,$$

having denoted the Fresnel integral (2.22) with $F_q(\varphi)$. Let us first demonstrate that $F_q(\varphi)$ is continuous: If $\varphi_k\to\varphi$, $F_q(\varphi_k)\to F_q(\varphi)$:

$$F_q(\varphi_k)-F_q(\varphi)=\int_{\mathscr{H}}\left(\exp\left\{-\frac{i}{2}[\Gamma'+\overline{\Gamma}_k,B(\Gamma'+\overline{\Gamma}_k)]\right\}\right.$$

$$\left.-\exp\left\{-\frac{i}{2}[\Gamma'+\overline{\Gamma},B(\Gamma'+\overline{\Gamma})]\right\}\right)$$

$$\times d\left(\chi*\exp\left[-\frac{i}{\hbar}\zeta\right]*\rho\right)(\Gamma'),$$

with

$$\overline{\Gamma}_k=\int_0^t\varphi_k(\tau)\frac{\Gamma_\tau^+-\Gamma_\tau^-}{2}\,d\tau$$

and

$$\overline{\Gamma}=\int_0^t\varphi(\tau)\frac{\Gamma_\tau^+-\Gamma_\tau^-}{2}\,d\tau.$$

Observe that $\overline{\Gamma}_k\underset{\mathscr{H}}{\to}\overline{\Gamma}$ if $\varphi_k\to\varphi$ (a uniform convergence of φ_k to φ is sufficient). Furthermore, it should be borne in mind that, since $\mathscr{H}$ is a complete separable metric space, any finite measure on it is tight[8]; hence, $\forall\epsilon$, there exists K such that

$$\int_{\|\Gamma'\|>K}d\left|\chi*\exp\left(-\frac{i}{\hbar}\zeta\right)*\rho\right|(\Gamma')<\epsilon.$$

Then

$$|F_q(\varphi_k) - F_q(\varphi)| \leqslant 2\epsilon + \int_{\|\Gamma'\|<K} [2|(\Gamma', B(\bar{\Gamma}_k - \bar{\Gamma}))| + |(\bar{\Gamma}_k, B\bar{\Gamma}_k) - (\bar{\Gamma}, B\bar{\Gamma})|] \times d|\chi * \exp\left(-\frac{i}{\hbar}\xi\right) * \rho|(\Gamma').$$

We conclude $F_q(\varphi_k) \to F_q(\varphi)$. Now

$$\lim_{k\to\infty} L(\varphi_k) = \lim_{k\to\infty} \exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t \varphi_k^2(\tau)\,d\tau\right] \times \int_R \exp\left[iq\int_0^t \varphi_k(\tau)\,d\tau\right] F_q(\varphi_k)\,dq = L(\varphi)$$

as long as it is possible to exchange the limit with the integral in q. Let us denote with $M_q(\varphi)$ the expression which gives $L(\varphi)$ when integrated in q:

$$M_q(\varphi) = \exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t \varphi^2(\tau)\,d\tau\right] \times \exp\left[iq\int_0^t \varphi(\tau)\,d\tau\right] F_q(\varphi). \tag{3.4}$$

Since $M_q(\varphi)$ is positive definite (as we will show in the proof of Proposition 3.3), $|M_q(\varphi)| \leqslant M_q(0)$ holds, and since we know that $\int_R M_q(0)\,dq = 1$ (normalization), this first shows that $\int_R M_q(\varphi)\,dq$ exists (this point was left open in the proof of Proposition 2.1), and, second (by the dominated convergence theorem), shows that it is possible to exchange the limit with the integral. ■

The positive definiteness of $L(\varphi)$ was more evident in the initial expression (1.5) than in expression (3.1) drawn from the calculation of the Fresnel integral. In order to regain an expression of a similar type to (1.5), we have to introduce a finite-dimensional approximation of the Feynman path integral. Let us take a Fresnel integral on $\tilde{\mathfrak{H}}$,

$$\widetilde{\int_{\tilde{\mathfrak{H}}}} \exp\left[\frac{i}{2}(\gamma,\gamma)_{\tilde{\mathfrak{H}}}\right] f(\gamma)\,d\gamma$$

with $f \in \mathcal{F}(\tilde{\mathfrak{H}})$ [i.e., $f = \int_{\tilde{\mathfrak{H}}} \exp[i(\gamma,\gamma')_{\tilde{\mathfrak{H}}}]\,d\mu_f(\gamma')$ with $\mu_f \in \mathcal{M}(\tilde{\mathfrak{H}})$]. As Truman has shown,[9] it is possible to give a rigorous definition of this integral, equivalent to that of Albeverio and Høegh-Krohn, in the following manner. When the interval $(0,t)$ has been subdivided in n equal intervals $(j\Delta t,(j+1)\Delta t)\, j = 0, 1, \ldots, n-1, \Delta t = t/n$, the following polygonal approximation of the trajectories $\gamma(\tau)$ is made:

$$(P_n,\gamma)(\tau) = \gamma_j + (\tau - j\Delta t)\Delta\gamma_j \Delta t^{-1}, \text{ for } j\Delta t \leqslant \tau < (j+1)\Delta t, \tag{3.5}$$

where we have taken $\gamma_j = \gamma(j\Delta t)$, $\Delta\gamma_j = \gamma_{j+1} - \gamma_j$. P_n is a projection in $\tilde{\mathfrak{H}}$, and one can see

$$(\gamma', P_n\gamma)_{\tilde{\mathfrak{H}}} = \frac{m}{\hbar}\sum_{j=0}^{n-1} \Delta\gamma_j' \Delta\gamma_j \Delta t^{-1}.$$

The following result holds. If $f \in \mathcal{F}(\tilde{\mathfrak{H}})$, the existence of the limit in the following expression is guaranteed, and we have, writing $f(P_n\gamma) = f(\gamma_0,\cdots,\gamma_{n-1})$,

$$\lim_{n\to\infty}\left(2\pi i\Delta t\frac{m}{\hbar}\right)^{-n/2}\int_{R^n}\exp\left[\frac{i}{2}\frac{m}{\hbar}\Delta t^{-1}\sum_{j=0}^{n-1}\Delta\gamma_j^2\right] \times f(\gamma_0,\cdots,\gamma_{n-1})\,d\gamma_0\cdots d\gamma_{n-1} = \widetilde{\int_{\tilde{\mathfrak{H}}}}\exp\left[\frac{i}{2}(\gamma,\gamma)_{\tilde{\mathfrak{H}}}\right] f(\gamma)\,d\gamma \overset{\text{def}}{=} \int_{\tilde{\mathfrak{H}}}\exp\left[-\frac{i}{2}(\gamma',\gamma')_{\tilde{\mathfrak{H}}}\right] d\mu_f(\gamma'). \tag{3.6}$$

Proposition 3.3: $L(\varphi)$ is a positive definite functional.

Proof: $L(\varphi) = \int_R dq\, M_q(\varphi)$, where $M_q(\varphi)$ is given by (3.4). It is sufficient to show that $M_q(\varphi)$ is positive definite for any q. Let us rewrite expression (3.4) making use of the polygonal approximation. In order to have a strictly ordered set of partitions of the interval $(0,t)$, let us take $n = 2^k$, $k = 1, 2, \cdots$. Because of the continuity of $M_q(\varphi)$ with respect to the topology of uniform convergence, we can restrict ourselves to considering the φ functions which are, for k sufficiently large, constant on the $n = 2^k$ intervals into which $(0,t)$ is subdivided. An obvious extension of Truman's result gives us the following:

$$M_q(\varphi) = \lim_{n\to\infty}\left(2\pi\Delta t\frac{m}{\hbar}\right)^{-n}\exp\left[iq\Delta t\sum_{j=0}^{n-1}\varphi_j\right]\int_{R^{2n}}\exp\left[\frac{i}{2}\frac{m}{\hbar}\Delta t^{-1}\sum_{j=0}^{n-1}(\Delta\gamma_j^{+2} - \Delta\gamma_j^{-2})\right] \times \psi_0(\gamma_0^+ + q)\gamma_0^*(\psi_0^- + q)\exp\left[-\frac{i}{\hbar}\int_0^t [V((P_n\gamma^+)(\tau) + q) - V((P_n\gamma^-)(\tau) + q)]\,d\tau\right] \times \exp\left[-\frac{\tilde{\alpha}}{4}\Delta t\sum_{j=0}^{n-1}(\bar{\gamma}_j^+ - \bar{\gamma}_j^-)^2\right]\exp\left[-\frac{\tilde{\alpha}}{48}\Delta t\sum_{j=0}^{n-1}(\Delta\gamma_j^+ - \Delta\gamma_j^-)^2\right]\cdot\left\{\exp\left(i\Delta t\sum_{j=0}^{n-1}\varphi_j\frac{\bar{\gamma}_j^+ + \bar{\gamma}_j^-}{2}\right) \times \exp\left(-\frac{1}{4\tilde{\alpha}}\Delta t\sum_{j=0}^{n-1}\varphi_j^2\right)\right\} d\gamma_0^+\cdots d\gamma_{n-1}^+\, d\gamma_0^-\cdots d\gamma_{n-1}^-, \tag{3.7}$$

where $(P_n\gamma^\pm)(\tau) = \gamma_j^\pm + (\tau - j\Delta t)\Delta\gamma_j^\pm \Delta t^{-1}$ for $j\Delta t \leqslant \tau < (j+1)\Delta t$, $\bar{\gamma}_j^\pm = \gamma_j^\pm + \frac{1}{2}\Delta\gamma_j^\pm$, $\varphi_j = \varphi(j\Delta t)$.

Let us take into account that the expression in curly brackets is the Fourier transform of a Gaussian; then

$$\exp\left[-\frac{\tilde{\alpha}}{4}\Delta t\sum_{j=0}^{n-1}(\bar{\gamma}_j^+ - \bar{\gamma}_j^-)^2\right]\left\{\exp\left(i\Delta t\sum_{j=0}^{n-1}\varphi_j\frac{\bar{\gamma}_j^+ + \bar{\gamma}_j^-}{2}\right)\exp\left(-\frac{1}{4\tilde{\alpha}}\Delta t\sum_{j=0}^{n-1}\varphi_j^2\right)\right\} = \exp\left[-\frac{\tilde{\alpha}}{4}\Delta t\sum_{j=0}^{n-1}(\bar{\gamma}_j^+ - \bar{\gamma}_j^-)^2\right]\left(\frac{\tilde{\alpha}\Delta t}{\pi}\right)^{n/2}\int_{R^n}\exp\left[i\Delta t\sum_{j=0}^{n-1}\varphi_j x_j - \tilde{\alpha}\Delta t\sum_{j=0}^{n-1}\left(x_j - \frac{\bar{\gamma}_j^+ + \bar{\gamma}_j^-}{2}\right)^2\right] dx_0\cdots dx_{n-1}$$

$$= \left(\frac{\tilde{\alpha}\Delta t}{\pi}\right)^{n/2} \int_{R^n} \exp\left(i\Delta t \sum_{j=0}^{n-1} \varphi_j x_j\right) \exp\left[-\frac{\tilde{\alpha}}{2}\Delta t \sum_{j=0}^{n-1} (x_j - \bar{\gamma}_j^{+})^2\right]$$
$$\times \exp\left[-\frac{\tilde{\alpha}}{2}\Delta t \sum_{j=0}^{n-1} (x_j - \bar{\gamma}_j^{-})^2\right] dx_0 \cdots dx_{n-1}.$$

We can now rewrite relation (3.7) in the following way:

$$M_q(\varphi) = \lim_{n\to\infty} \lim_{L\to\infty} \left(2\pi\Delta t \frac{m}{\hbar}\right)^{-n} \left(\frac{\tilde{\alpha}\Delta t}{\pi}\right)^{n/2} \int_{R^n} dx_0 \cdots dx_{n-1} \exp\left[i\Delta t \sum_{j=0}^{n-1} \varphi_j (x_j + q)\right]$$
$$\times \int_{-L}^{+L} d\gamma_0^{+} \cdots \int_{-L}^{+L} d\gamma_{n-1}^{+} \int_{-L}^{+L} d\gamma_0^{-} \cdots \int_{-L}^{+L} d\gamma_{n-1}^{-} \exp\left[\frac{i}{2}\frac{m}{\hbar}\Delta t^{-1} \sum_{j=0}^{n-1} (\Delta\gamma_j^{+2} - \Delta\gamma_j^{-2})\right] \psi(\gamma_0^{+} + q)\psi^*(\gamma_0^{-} + q)$$
$$\times \exp\left\{-\frac{i}{\hbar}\int_0^t [V((P_n\gamma^{+})(\tau) + q) - V((P_n\gamma^{-})(\tau) + q)]\, d\tau\right\} \exp\left[-\frac{\tilde{\alpha}}{2}\Delta t \sum_{j=0}^{n-1} (x_j - \bar{\gamma}_j^{+})^2\right]$$
$$\times \exp\left[-\frac{\tilde{\alpha}}{2}\Delta t \sum_{j=0}^{n-1} (x_j - \bar{\gamma}_j^{-})^2\right] \exp\left[-\frac{\tilde{\alpha}}{48}\Delta t \sum_{j=0}^{n-1} (\Delta\gamma_j^{+} - \Delta\gamma_j^{-})^2\right], \tag{3.8}$$

and noting again that

$$\exp\left[-\frac{\tilde{\alpha}}{48}\Delta t \sum_{j=0}^{n-1} (\Delta\gamma_j^{+} - \Delta\gamma_j^{-})^2\right] = \left(\frac{12}{\tilde{\alpha}\Delta t \pi}\right)^{n/2} \int_{R^n} \exp\left[-\frac{12}{\tilde{\alpha}\Delta t} \sum_{j=0}^{n-1} K_j^2 + i \sum_{j=0}^{n-1} K_j (\Delta\gamma_j^{+} - \Delta\gamma_j^{-})\right] dK_0 \cdots dK_n,$$

we obtain finally

$$M_q(\varphi) = \lim_{n\to\infty} \lim_{L\to\infty} \int_{R^n} \exp\left[i\Delta t \sum_{j=0}^{n-1} \varphi_j (x_j + q)\right] P_L(x_0, \cdots, x_{n-1}) dx_0 \cdots dx_{n-1},$$

where $P_L(x_0, \cdots, x_{n-1})$ is a positive function.

It is now clear that if we take k functions $\varphi^{(i)}$ which, for a sufficiently large n, are constant on the n intervals of the subdivision of $(0,t)$, and if α_i are k complex constants, then

$$\sum_{i,l=1}^{k} \alpha_i \alpha_l^* M_q(\varphi^{(i)} - \varphi^{(l)}) = \lim_{n\to\infty} \lim_{L\to\infty} \int P_L(x_0, \cdots, x_{n-1}) \sum_{i,l=1}^{k} \alpha_i \bar{\alpha}_l^* \exp\left[i\Delta t \sum_{j=0}^{n-1} (\varphi_j^{(i)} - \varphi_j^{(l)})(x_j + q)\right] dx_0 \cdots dx_{n-1} \geqslant 0. \blacksquare$$

4. STRUCTURE OF THE PROCESS

We can write $L(\varphi)$ in such a way as to permit us to analyze the stochastic processes in terms of simpler "processes." Not all these processes are true stochastic processes, because in this analysis some "probabilities" occur which may be negative. However, it seems to us that, particularly in the free case, the result is sufficiently simple to throw light on the nature of the process we are studying here. Taking into account that

$$\int_R \exp\left[iq\left(\alpha - \alpha' + \sum_{i=1}^{n} \beta_i + \int_0^t \varphi(\tau)\, d\tau\right)\right] dq = 2\pi\delta\left(\alpha - \alpha' + \sum_{i=1}^{n} \beta_i + \int_0^t \varphi(\tau)\, d\tau\right),$$

let us rewrite relation (3.1):

$$L(\varphi) = \exp\left[-\frac{1}{4\tilde{\alpha}} \int_0^t \varphi^2(\tau)\, d\tau\right] \int_{R^2} d\nu(\alpha)\, d\nu^*(\alpha') \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{R^n} \prod_{i=1}^{n} d\tau_i \int_{R^n} \prod_{i=1}^{n} d\mu(\beta_i)$$
$$\times 2\pi\delta\left(\alpha - \alpha' + \sum_{i=1}^{n} \beta_i + \int_0^t \varphi(\tau)\, d\tau\right) \exp\left\{i\frac{\hbar}{m}\frac{\alpha + \alpha'}{2}\left[\sum_{i=1}^{n} \beta_i \tau_i + \int_0^t \varphi(\tau)\tau\, d\tau\right]\right\}$$
$$\times \exp\left[-\frac{\tilde{\alpha}}{4} \int_0^t \frac{\hbar^2}{m^2} \left(\sum_{\tau_l > \tau} \beta_l (\tau_l - \tau) + \int_\tau^t \varphi(\sigma)(\sigma - \tau)\, d\sigma\right)^2 d\tau\right]$$
$$\times \prod_{i=1}^{n} \left(\exp\left[i\frac{\hbar}{m}\frac{\beta_i}{2}\left(\sum_{\tau_l > \tau_i} \beta_l (\tau_l - \tau_i) + \int_{\tau_i}^t \varphi(\tau)(\tau - \tau_i)\, d\tau\right)\right] - \text{c.c.}\right). \tag{4.1}$$

First of all let us consider the first term of this series, which gives the characteristic functional in the free case, $L_{V=0}(\varphi)$. To obtain familiar expressions, and to use variables with the dimensions of a momentum, we take $\alpha = P/\hbar$, $d\nu(\alpha) = (2\pi\hbar)^{-1/2}\tilde{\psi}_0(P)\, dP$, so that

$$\psi_0(x) = \int_R \exp(i\alpha x)\, d\nu(\alpha) = \int_R (2\pi\hbar)^{-1/2} \exp\left(\frac{i}{\hbar} Px\right) \tilde{\psi}_0(P)\, dP.$$

Analogously $\alpha' = P'/\hbar$. Then we introduce the variables $p = (P + P')/2$, $\pi = P - P'$ and we integrate on π. Lastly, also taking into account that

$$-\int_0^t \varphi(\sigma)\tau\, d\sigma + \int_0^t \varphi(\sigma)(\tau \vee \sigma)\, d\sigma = \int_\tau^t \varphi(\sigma)(\sigma - \tau)\, d\sigma,$$

we obtain

$$L_{V=0}(\varphi) = \exp\left[-\frac{1}{4\tilde{\alpha}} \int_0^t \varphi^2(\tau)\, d\tau\right]$$

$$\times\exp\left[-\frac{\tilde{\alpha}}{4}\int_0^t\left(\frac{\hbar}{m}\int_\tau^t\varphi(\sigma)(\sigma-\tau)\,d\sigma\right)^2 d\tau\right]$$
$$\times\int_R\tilde{\psi}_0\left(p-\frac{\hbar}{2}\int_0^t\varphi(\tau)\,d\tau\right)$$
$$\times\tilde{\psi}_0^*\left(p+\frac{\hbar}{2}\int_0^t\varphi(\tau)\,d\tau\right)$$
$$\times\exp\left[i\frac{p}{m}\int_0^t\varphi(\tau)\tau\,d\tau\right]dp. \tag{4.2}$$

If we introduce the Wigner distribution associated with the initial state $|\psi_0\rangle$,[10]

$$F(q,p)=\frac{1}{2\pi}\int_R\tilde{\psi}_0\left(p-\frac{\hbar}{2}\theta\right)\tilde{\psi}_0^*\left(p+\frac{\hbar}{2}\theta\right)\exp(-i\theta q)\,d\theta, \tag{4.3}$$

we get

$$L_{V=0}(\varphi)=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]$$
$$\times\int_{R^2}F(q,p)\exp\left[i\int_0^t\varphi(\tau)\right.$$
$$\left.\times\left(q+\frac{p}{m}\tau\right)d\tau\right]dq\,dp$$
$$\times\exp\left\{-\frac{\tilde{\alpha}}{4}\left(\frac{\hbar}{m}\right)^2\right.$$
$$\left.\times\int_0^t\left[\int_\tau^t\varphi(\sigma)(\sigma-\tau)\,d\sigma\right]^2 d\tau\right\}. \tag{4.4}$$

This factorization of $L_{V=0}(\varphi)$ indicates that the process is made up of three simple processes. The first is a white noise, restricted to the interval $(0,t)$, which, as is known, is a stationary generalized stochastic process with independent values at every point. It is responsible for the fact that the process cannot be ordinary; the presence of this factor is essential for positivity. The other two are ordinary processes. The second process is the superposition with initial distribution $F(q,p)$ of the deterministic processes concentrated on the trajectories $x_{\text{free}}(\tau)=q+(p/m)\tau$; it is not a true process because $F(q,p)$ can take negative values. Lastly, the presence of the factor

$$\exp\left\{-\frac{\tilde{\alpha}}{4}\left(\frac{\hbar}{m}\right)^2\int_0^t\left[\int_\tau^t\varphi(\sigma)(\sigma-\tau)\,d\sigma\right]^2 d\tau\right\}=D(\varphi)$$

indicates that the velocity of the particle undergoes a Brownian motion caused by the continual observations of the position. This can be seen by observing that the derivative of such a functional is

$$D'(\varphi)\overset{\text{def}}{=}D(-\dot{\varphi})$$
$$=\exp\left[-\frac{\tilde{\alpha}}{4}\frac{\hbar^2}{m^2}\int_0^t\left(\int_\tau^t\varphi(\sigma)d\sigma\right)^2 d\tau\right]:$$

This is the characteristic functional of a Wiener process in the interval $(0,t)$ (see Refs. 5 or 6). For example, denoting by $p(v_1,\ldots,v_n)$ the density of the joint probability distribution at n times $t_1,\ldots,t_n$, we have

$$p(v_1,..v_n)=(\tilde{\alpha}\pi)^{-n/2}[t_1(t_2-t_1)\cdots(t_n-t_{n-1})]^{-1/2}\left(\frac{m}{\hbar}\right)^n$$
$$\times\exp\left[-\frac{1}{\tilde{\alpha}}\frac{m^2}{\hbar^2}\right.$$
$$\left.\times\left(\frac{v_1^2}{t_1}+\frac{(v_2-v_1)^2}{t_2-t_1}+\cdots+\frac{(v_n-v_{n-1})^2}{t_n-t_{n-1}}\right)\right], \tag{4.5}$$

which shows that the larger $\tilde{\alpha}$ is, i.e., the greater the accuracy of the Gaussian instruments, the greater is the dispersion introduced by this factor. Let us indicate by $[(\Omega,\Sigma,W);B_\tau(\omega)]$ a realization of this process: (Ω,Σ,W) is the probability space; $B_\tau(\omega)$, measurable functions on (Ω,Σ), are the random variables which here have the meaning of velocity of the particle at time τ. If we also introduce the random variables $\xi_\tau(\omega)=\int_0^\tau B_\tau(\omega)\,d\tau$ with the meaning of position at time τ, the following representation of $L_{V=0}(\varphi)$ holds:

$$L_{V=0}(\varphi)$$
$$=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_{R^2}dq\,dp\,F(q,p)\int_\Omega dW(\omega)$$
$$\times\exp\left\{i\int_0^t\varphi(\tau)\left[q+\frac{p}{m}\tau+\xi_\tau(\omega)\right]d\tau\right\} \tag{4.6}$$

In the presence of a potential the following result holds, for which the calculations are to be found in Appendix B:

$$L(\varphi)$$
$$=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_{R^2}dq\,dp\,F(q,p)\int_\Omega dW(\omega)\sum_{n=0}^\infty$$
$$\times\int_0^t d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots\int_0^{\tau_2}d\tau_1\int_{R^n}\prod_{j=1}^n d\pi_j$$
$$\times\prod_{i=1}^n\left[\frac{4}{\hbar}\operatorname{Im}\left(\langle\pi_i|\hat{V}(x+X_{\tau_i}(q,p,J,\omega))|-\pi_i\rangle\right)\right]$$
$$\times\exp\left[i\int_0^t\varphi(\tau)X_\tau(q,p,J,\omega)\,d\tau\right], \tag{4.7}$$

where

$$X_\tau(q,p,J,\omega)=q+\frac{p}{m}\tau+\sum_{\tau_l<\tau}\frac{\pi_l}{m}(\tau-\tau_l)+\xi_\tau(\omega)$$

and J denote the set $\{(\pi_l,\tau_l);\ l \text{ such that } \tau_l<\tau\}$.

The presence of the potential causes a "jump process" for the velocity of the particle to appear. (Jump processes have been discussed in connection with Feynman path integrals by Maslov and other authors, see Ref. 11.) The "probability density" for a jump in velocity of π_i/m at the time τ_i is given by the imaginary part of the matrix element of the potential, translated of the quantity $X_{\tau_i}(q,p,J,\omega)$, between the states of definite momentum $|\pi_i\rangle$ and $|-\pi_i\rangle$. We have given this representation of $L(\varphi)$ because of its formal simplicity, but we should like to stress the "quantum nature" of this jump process. The appearance here of negative "probabilities" is crucial, as is shown by the fact that the total "probability" of the particle making n jumps in velocity, $n\neq0$, is zero. (This is demonstrated by the argument given in the proof of Proposition 3.1).

In conclusion, let us observe that it should be possible to treat the case of the harmonic oscillator, or more generally of an anharmonic oscillator [$V(x)=\frac{1}{2}m\omega^2x^2+U(x)$ with $U(x)$ Fourier transform of a bounded complex measure], in a simi-

lar way. We expect the main result of this paper, i.e., the existence of the stochastic process, still to hold; the analogous relations of (4.6) and (4.7) should contain the trajectories $x_{\text{h.o.}}(\tau) = q\cos(\omega\tau) + (p/m\omega)\sin(\omega\tau)$ in place of the free trajectories $x_{\text{free}}(\tau) = q + (p/m)\tau$. This case will be dealt with in a future work, in which we hope to discuss also the case of a boson field theory; this could represent an intermediate step toward an objective description of a macrosystem. As we have said in the Introduction, this is in fact the final aim of the present research.

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APPENDIX A

Consider $\mathcal{F}(\bar{x},\Delta t)$, which is defined by (1.2)–(1.4). We have

$$\delta\left(\bar{x} - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \int_R \frac{\Delta t}{2\pi}\exp\left[-iK\Delta t\left(\bar{x} - \frac{1}{n}\sum_{j=1}^{n} x_j\right)\right] dK$$

and

$$\int_R \exp\left[\frac{iK\Delta t}{n} x_j\right] Q\left(x_j; \frac{\tilde{\alpha}\Delta t}{n}\right)\hat{\rho}\left(\frac{\tilde{\alpha}\Delta t}{\pi n}\right)^{1/2} dx_j = \tilde{Q}_K\left(\tilde{\alpha}; \frac{\Delta t}{n}\right)\hat{\rho}, \tag{A1}$$

where

$$\tilde{Q}_K\left(\tilde{\alpha}; \frac{\Delta t}{n}\right) = \exp\left[\left(-\frac{\tilde{\alpha}}{4}[\hat{q},[\hat{q},\cdot]] + \frac{i}{2}K\{\hat{q},\cdot\} - \frac{K^2}{4\tilde{\alpha}}\right)\frac{\Delta t}{n}\right] \tag{A2}$$

(see below for the proof). Hence

$$\mathcal{F}(\bar{x},\Delta t) = \lim_{n\to\infty}\int_R \frac{\Delta t}{2\pi}\exp(-iK\Delta t\bar{x}) \times\left[\tilde{Q}_K\left(\tilde{\alpha}; \frac{\Delta t}{n}\right)U\left(\frac{\Delta t}{n}\right)\right]^n dK,$$

and, by the Lie–Kato–Trotter product formula, relation (2.1) should follow.

Proof of relation (A 1):

$$\int_R \exp\left(\frac{iK\Delta t}{n}x\right)\left\langle q\left|Q\left(x; \frac{\tilde{\alpha}\Delta t}{n}\right)\hat{\rho}\right|q'\right\rangle\left(\frac{\tilde{\alpha}\Delta t}{n\pi}\right)^{1/2} dx$$

$$= \int_R \exp\left\{\frac{iK\Delta t}{n}x - \frac{1}{2}\tilde{\alpha}\frac{\Delta t}{n}[(q-x)^2 + (q'-x)^2]\right\} \times\langle q|\hat{\rho}|q'\rangle\left(\frac{\tilde{\alpha}\Delta t}{n\pi}\right)^{1/2} dx$$

$$= \exp\left[-\frac{1}{2}\tilde{\alpha}\frac{\Delta t}{n}(q^2 + q'^2)\right]\langle q|\hat{\rho}|q'\rangle \times\int_R \exp\left\{-\tilde{\alpha}\frac{\Delta t}{n}x^2 + \left[\tilde{\alpha}\frac{\Delta t}{n}(q+q') + iK\frac{\Delta t}{n}\right]x\right\}\left(\frac{\tilde{\alpha}\Delta t}{n\pi}\right)^{1/2} dx$$

$$= \exp\left[-\frac{1}{2}\tilde{\alpha}\frac{\Delta t}{n}(q^2 + q'^2)\right]\langle q|\hat{\rho}|q'\rangle \times\exp\left\{\frac{n}{4\tilde{\alpha}\Delta t}\left[\tilde{\alpha}\frac{\Delta t}{n}(q+q') + iK\frac{\Delta t}{n}\right]^2\right\}$$

$$= \exp\left[\frac{\Delta t}{n}\left(-\frac{1}{2}\tilde{\alpha}(q^2+q'^2) + \frac{\tilde{\alpha}}{4}(q^2+q'^2+2qq') + \frac{i}{2}K(q+q') - \frac{K^2}{4\tilde{\alpha}}\right)\right]\langle q|\hat{\rho}|q'\rangle$$

$$= \exp\left[\frac{\Delta t}{n}\left(-\frac{\tilde{\alpha}}{4}(q^2+q'^2-2qq') + \frac{i}{2}K(q+q') - \frac{K^2}{4\tilde{\alpha}}\right)\right]\langle q|\hat{\rho}|q'\rangle$$

$$= \left\langle q\left|\exp\left[\frac{\Delta t}{n}\left(-\frac{\tilde{\alpha}}{4}[\hat{q},[\hat{q},\cdot]] + \frac{i}{2}K\{\hat{q},\cdot\} - \frac{K^2}{4\tilde{\alpha}}\right)\right]\hat{\rho}\right|q'\right\rangle$$

$$= \left\langle q\left|\tilde{Q}_K\left(\tilde{\alpha}; \frac{\Delta t}{n}\right)\hat{\rho}\right|q'\right\rangle.$$

APPENDIX B

With a technique analogous to the one we used in the free case and with the same notations, we now obtain

$$L(\varphi) = \exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t \varphi^2(\tau)\,d\tau\right]\int_{R^2} dq\,dp\,F(q,p) \times\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\int_{R^n}\prod_{i=1}^{n} d\tau_i\int_{R^n}\prod_{i=1}^{n} d\mu(\beta_i) \times\exp\left[iq\left(\sum_{i=1}^{n}\beta_i + \int_0^t \varphi(\tau)\,d\tau\right)\right] \times\exp\left\{i\frac{p}{m}\left[\sum_{i=1}^{n}\beta_i\tau_i + \int_0^t \varphi(\tau)\tau\,d\tau\right]\right\} \times\exp\left\{-\frac{\tilde{\alpha}}{4}\int_0^t\left(\frac{\hbar}{m}\right)^2\left[\sum_{\tau_l>\tau}\beta_l(\tau_l-\tau) + \int_\tau^t \varphi(\sigma)(\sigma-\tau)\,d\sigma\right]^2 d\tau\right\} \times\prod_{i=1}^{n}\left(\exp\left[i\frac{\hbar}{m}\frac{\beta_i}{2}\left(\sum_{\tau_l>\tau_i}\beta_l(\tau_l-\tau_i) + \int_{\tau_i}^t \varphi(\tau)(\tau-\tau_i)\,d\tau\right)\right] - \text{c.c.}\right). \tag{B1}$$

Because of the reality of $V(x)$, $d\mu(-\beta) = d\mu^*(\beta)$ holds. We can rewrite (B1) as follows:

$$L(\varphi) = \exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t \varphi^2(\tau)\,d\tau\right]\int_{R^2} dq\,dp\,F(q,p) \times\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\sum_{p=0}^{n} p!(n-p)!(-)^{n-p} \times\int_{R^p}\prod_{j=1}^{p} d\tau_j\int_{R^{n-p}}\prod_{h=1}^{n-p} d\tau'_h\int_{R^p}\prod_{j=1}^{p} d\mu(\beta_j) \times\int_{R^{n-p}}\prod_{h=1}^{n-p} d\mu^*(\beta'_h)$$

$$\times\exp\left\{iq\left[\sum_{j=1}^{p}\beta_j-\sum_{h=1}^{n-p}\beta'_h+\int_0^t\varphi(\tau)d\tau\right]\right\}$$
$$\times\exp\left\{i\frac{p}{m}\left[\sum_{j=1}^{p}\beta_j\tau_j-\sum_{h=1}^{n-p}\beta'_h\tau'_h+\int_0^t\varphi(\tau)\,d\tau\right]\right\}$$
$$\times\exp\left\{i\frac{\hbar}{m}\sum_{j=1}^{p}\frac{\beta_j}{2}\left[\sum_{\tau_l>\tau_j}\beta_l(\tau_l-\tau_j)\right.\right.$$
$$\left.\left.-\sum_{\tau'_r>\tau_j}\beta'_r(\tau'_r-\tau_j)+\int_{\tau_j}^t\varphi(\tau)(\tau-\tau_j)\,d\tau\right]\right\}$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{h=1}^{n-p}\frac{\beta'_h}{2}\left[\sum_{\tau_l>\tau'_h}\beta_l(\tau_l-\tau'_h)\right.\right.$$
$$\left.\left.-\sum_{\tau'_r>\tau'_h}\beta'_r(\tau'_r-\tau'_h)+\int_{\tau'_h}^t\varphi(\tau)(\tau-\tau'_h)\,d\tau\right]\right\}$$
$$\times\int_\Omega dW(\omega)\exp\left[i\int_0^t\varphi(\tau)\xi_\tau(\omega)\,d\tau\right.$$
$$\left.+\sum_{j=1}^{p}\beta_j\xi_{\tau_j}(\omega)-\sum\beta'_h\xi_{\tau'_h}(\omega)\right].$$

After rearranging the double summations in the following way,

$$\sum_{j=1}^{p}\frac{\beta_j}{2}\left[\sum_{\tau_l>\tau_j}\beta_l(\tau_l-\tau_j)-\sum_{\tau'_r>\tau_j}\beta'_r(\tau'_r-\tau_j)\right]$$
$$+\sum_{h=1}^{n-p}\frac{\beta'_h}{2}\left[\sum_{\tau_l>\tau'_h}\beta_l(\tau_l-\tau'_h)-\sum_{\tau'_r>\tau'_h}\beta'_r(\tau'_r-\tau'_h)\right]$$
$$=\sum_{j=1}^{p}\frac{\beta_j}{2}\left[\sum_{\tau_l<\tau_j}\beta_l(\tau_j-\tau_l)+\sum_{\tau'_r<\tau_j}\beta'_r(\tau-\tau'_r)\right]$$
$$-\sum_{h=1}^{n-p}\frac{\beta'_h}{2}\left[\sum_{\tau_l<\tau'_h}\beta_l(\tau'_h-\tau_l)+\sum_{\tau'_r<\tau'_h}\beta'_r(\tau'_h-\tau'_r)\right],$$

we finally obtain

$$L(\varphi)=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_{R^2}dq\,dp\,F(q,p)$$
$$\times\int_\Omega dW(\omega)\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\int_{R^n}\prod_{i=1}^{n}d\tau_i$$
$$\times\int_{R^n}\prod_{i=1}^{n}\left(d\mu(\beta_i)\exp\left\{i\beta_i\left[q+\frac{p}{m}\tau_i\right.\right.\right.$$
$$\left.\left.\left.+\sum_{\tau_l<\tau_i}\frac{\hbar}{m}\frac{\beta_l}{2}(\tau_i-\tau_l)+\xi_{\tau_i}(\omega)\right]\right\}-\text{c.c.}\right)$$
$$\times\exp\left\{i\int\varphi(\tau)\left[q+\frac{p}{m}\tau\right.\right.$$
$$\left.\left.+\sum_{\tau_j<\tau}\frac{\hbar}{m}\frac{\beta_j}{2}(\tau-\tau_j)+\xi_\tau(\omega)\right]d\tau\right\}. \qquad \text{(B2)}$$

Defining the random variables

$$X_\tau(q,p,J,\omega)=q+\frac{p}{m}\tau+\sum_{\tau_l<\tau}\frac{\pi_l}{m}(\tau-\tau_l)+\xi_\tau(\omega),$$

where we have taken $\pi=\frac{1}{2}\hbar\beta$, $J=\{(\pi_l,\tau_l);\ l \text{ such that } \tau_l<\tau\}$, and observing that (we assume μ absolutely continuous with respect to the Lebesgue measure)

$$\exp\left\{i\beta_i\left[q+\frac{p}{m}\tau_i+\sum_{\tau_l<\tau_i}\frac{\hbar}{m}\frac{\beta_l}{2}(\tau_i-\tau_l)+\xi_{\tau_i}(\omega)\right]\right\}d\mu(\beta_i)$$
$$=2\left(\frac{1}{2\pi\hbar}\int_R V(x+X_{\tau_i}(q,p,J,\omega))\right.$$
$$\left.\cdot\exp\left(-i\frac{2\pi_i}{\hbar}x\right)dx\right]d\pi_i,$$

we can rewrite (B2) in the following way:

$$L(\varphi)=\exp\left[-\frac{1}{4\tilde{\alpha}}\int_0^t\varphi^2(\tau)\,d\tau\right]\int_{R^2}dq\,dp\,F(q,p)$$
$$\times\int_\Omega dW(\omega)\sum_{n=0}^{\infty}\int_0^t d\tau_n\int_0^{\tau_n}d\tau_{n-1}\cdots\int_0^{\tau_2}d\tau_1$$
$$\times\int_{R^n}\prod_{j=1}^{n}d\pi_j\cdot\prod_{i=1}^{n}\left[\frac{4}{\hbar}\,\mathrm{Im}\left[\frac{1}{2\pi\hbar}\int_R V(x\right.\right.$$
$$\left.\left.+X_{\tau_i}(q,p,J,\omega))\exp\left(-i\frac{2\pi_i x}{\hbar}\right)dx\right]\right]$$
$$\times\exp\left[i\int_0^t\varphi(\tau)X_\tau(q,p,J,\omega)\,d\tau\right]. \qquad \text{(B3)}$$

[1] A. Barchielli, L. Lanz, and G. M. Prosperi, "A Model for Macroscopic Description and Continual Observations in Quantum Mechanics," Nuovo Cimento B **72**, 79–121 (1982).

[2] A. Daneri, A. Loinger, and G. M. Prosperi, "Quantum Theory of Measurement and Ergodicity Conditions," Nucl. Phys. **33**, 297–319 (1962).

[3] G. Ludwig, "Measuring and Preparing Processes," in *Lecture Notes in Physics* (Springer-Verlag, Berlin, 1973), Vol. 29, p. 122–162.

[4] E. B. Davies, "On the Repeated Measurement of Continuous Observables in Quantum Mechanics," J. Funct. Anal. **6**, 318–46 (1970).

[5] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1964), Vol. 4.

[6] T. Hida, *Stationary Stochastic Processes* (Princeton U.P., Princeton, NJ, 1970).

[7] S. A. Albeverio and R. J. Høegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Mathematics, Vol. 523 (Springer-Verlag, Berlin, 1976).

[8] K. R. Parthasarathy, *Probability Measures on Metric Spaces* (Academic, New York, 1967).

[9] A. Truman, "The Feynman maps and the Wiener integrals," J. Math. Phys. **19**, 1742–50 (1978).

[10] J. E. Moyal, "Quantum Mechanics as a Statistical Theory," Proc. Cambridge Phil. Soc. **45**, 99–124 (1949).

[11] A. M. Chebotarev and V. P. Maslov, "Processus de sauts et leurs applications dans la mécanique quantique," in *Lecture Notes in Physics* (Springer-Verlag, Berlin, 1979), Vol. 106, pp. 58–72; Ph. Combe, R. Rodriguez, R. Høegh-Krohn, M. Sirugue, and M. Sirugue-Collin, "Generalized Poisson processes in quantum mechanics and field theory," Phys. Rep. 77, 221–34 (1981).

Operators for the two-dimensional harmonic oscillator in an angular momentum basis

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The forms of the operators $\nu^\dagger$, ν, $\lambda^\dagger$, λ, which enable one to write the Hamiltonian of the two-dimensional isotropic harmonic oscillator in the form $H = \hbar\omega(2\nu^\dagger\nu + \lambda^\dagger\cdot\lambda + 1)$, are presented. Here $\nu^\dagger$ and ν are, respectively, the raising and lowering operators for $\nu^\dagger\nu$, the "radial" quantum number operator, while $\lambda^\dagger$ and λ are, respectively, the raising and lowering operators for M, the magnitude of the angular momentum operator. Corresponding to this decomposition of H in the angular momentum basis are the energy eigenvalues $E_{km} = \hbar\omega(2k + |m| + 1)$ with $k = 0, 1, 2, \cdots$ and $m = 0, \pm 1, \pm 2, \cdots$. Here k is a "radial" quantum number, and m is a "magnetic" quantum number. The commutation relations satisfied by the operators $\nu^\dagger$, ν, $\lambda^\dagger$, and λ are also presented.

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1. INTRODUCTION

Bracken and Leemon[1] have developed an algebraic theory of the three-dimensional isotropic harmonic oscillator in an angular momentum basis and have shown that the Hamiltonian H of this oscillator can be written in the form $H = \hbar\omega(2\nu^\dagger\nu + \lambda^\dagger\cdot\lambda + \frac{3}{2})$, where $\nu^\dagger$ and ν are, respectively, the raising and lowering operators for $\nu^\dagger\nu$, while $\lambda^\dagger$ and λ are the corresponding shift operators for $\lambda^\dagger\cdot\lambda$. The aim of this paper is to show that an analogous algebraic treatment can be represented for the two-dimensional isotropic harmonic oscillator.

It should be mentioned that the two-dimensional isotropic harmonic oscillator was long ago treated in an occupational number representation (see Chap. XII, Sec. 14 of Ref. 2). Even though raising and lowering operators were constructed for the angular momentum, both the energy and angular momentum eigenvalues were obtained, respectively, as sums and differences of the number of quanta of the positive and negative types. It will be shown that H can be written in the form $\hbar\omega(\nu^\dagger\nu + \lambda^\dagger\cdot\lambda + 1)$, where $\nu^\dagger$ and ν are, respectively, the raising and lowering operators for $\nu^\dagger\nu$, the "radial" quantum number operator, while $\lambda^\dagger$ and λ are, respectively, the raising and lowering operators for $\lambda^\dagger\cdot\lambda = M$, the magnitude of the angular momentum operator. The forms of these operators and the various commutation relations satisfied by them will be derived. In a subsequent paper, these operators will be used to construct the common eigenvectors of H and M.

Correspondingly, the eigenvalues of the Hamiltonian, when evaluated in an "angular momentum basis" (e.g., in polar coordinates, in the coordinate representation), appear in the form

$$E_{km} = \hbar\omega(2k + |m| + 1) \quad k = 0,1,2,\cdots, m = 0, \pm 1, \pm 2, \cdots.$$

Here k is a "radial" quantum number, and m is the "magnetic" quantum number. (The angular momentum in the plane is $\hbar m$.)

One would expect some differences to exist in the two-dimensional situation for the following reasons: (i) the group SO(2), according to whose representation the eigenvectors of the operators associated with the two-dimensional oscillator transform, is an abelian Lie group while SO(3) is nonabelian, and (ii) while L is an SO(3) scalar in the three-dimensional case, it is an SO(2) pseudoscalar in the two-dimensional situation. Consequently, the raising and lowering operators have different forms in the two-dimensional situation. In fact, since the angular momentum operator, for example, occurs in the denominator of expressions defining various operators and this angular momentum operator has both zero and the negative integers in its eigenvalue spectrum it becomes imperative to work with the magnitude of the angular momentum operator.

2. THE OPERATORS AND THEIR COMMUTATION RELATIONS

The forms of the operators ν, $\nu^\dagger$, λ, and $\lambda^\dagger$, which enable one to write the Hamiltonian of the two-dimensional oscillator in the form $\hbar\omega(2\nu^\dagger\nu + \lambda^\dagger\cdot\lambda + 1)$, will be presented. The angular momentum operator L of the two-dimensional oscillator is the following dimensionless SO(2) pseudoscalar

$$L = \tfrac{1}{2}\epsilon_{ij}l_{ij} \quad (i,j = 1,2)$$

$$l_{ij} = (x_i p_j - x_j p_i)/\hbar = i(a_i a_j^\dagger - a_j a_i^\dagger).$$

Here ϵ_{ij} is the totally antisymmetric tensor, the Levi–Civita symbol. Of course, l_{ij} has only one independent component l_{12}, and the definition actually reduces to $L = l_{12}$. Here, also, a_i and $a_i^\dagger$ $(i = 1, 2)$ are, respectively, the lowering and raising operators of the occupation number eigenvectors. They obey the usual commutation relations

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger], \tag{1a}$$

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{1b}$$

This L is the only generator of rotation for the SO(2) group. The same symbol L will be used to denote the operator corresponding to this infinitesimal group generator in the corresponding simple abelian Lie algebra.

Since the eigenvalue spectrum of L contains both the negative and positive integers and zero, and since this angular momentum operator occurs in the denominator of many

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of the operator identities to be given, it is necessary to replace L by its magnitude $M = |L|$. Suppose $\{\Phi_m\}$ is a complete set of eigenvectors of the operator L, then, for any vector Φ in the relevant Hilbert space,

$$L\Phi = \sum_m m c_m \Phi_m,$$

where m is the eigenvalue of L corresponding to the eigenvector Φ_m. M is simply defined by setting it equal to that operator which has the eigenvalue $|m|$ whenever L has the eigenvalue m. Thus

$$M\Phi = \sum_m |m| c_m \Phi_m$$

for Φ is the domain of L and M. It should be noted that M, unlike L, is a true scalar. It is clear that, for any eigenvector possessing the eigenvalue $m \geqslant 0$, M coincides with L. All the relations in this paper will be proved for this situation in which M is identical with L.

One can readily see that

$$M^2 = \tfrac{1}{2} l_{ij} l_{ij} = L^2,$$

$$[M, l_{ij}] = 0.$$

As presented in a paper by Bracken and Green,[3] the operators corresponding to the generators of the unitary irreducible representation of SO(n), can be viewed as elements of an anti-Hermitian matrix $[m]$ of operators. This matrix satisfies a polynomial equation, which is factorizable and which can be projected into the invariant eigenspaces of the representation. For the SO(2) group, this matrix polynomial is

$$F_2([m]) = [m^2] - M^2 = ([m] + M)([m] - M) = 0.$$

In this equation, M^2 denotes the operator M^2 multiplied by the unit matrix. The projection operators corresponding to this polynomial are

$$P^{(+)} = (M + [m])/2M,$$

$$P^{(-)} = (M - [m])/2M.$$

We can therefore define the following operators:

$$a_j^{(\pm)} = a_i (M\delta_{ij} \pm m_{ij})/2M \quad (j = 1,2), \tag{2}$$

where

$$[m_{ij}] = \begin{pmatrix} 0 & -iM \\ iM & 0 \end{pmatrix}$$

with $m_{ij} = a_i a_j^\dagger - a_j a_i^\dagger$.

Since Eq. (2) is ill defined on any Hilbert space vector which is an eigenvector of M with eigenvalue 0, we may modify the definition on such vectors to

$$\mathbf{a}^{(+)} = \mathbf{a}, \quad \mathbf{a}^{(-)} = \mathbf{0}.$$

An alternative definition of the 2-vector $\mathbf{a}$ can be given. Let us first note that

$$(l_{ij} + iM\delta_{ij})(l_{jk} - iM\delta_{jk}) = 0.$$

Hence, the following definition of $a_j^{(\pm)}$ can be given:

$$a_j^{(\pm)} = a_i (l_{ij} \pm iM\delta_{ij})/(\pm 2iM).$$

On substituting l_{ij} into this equation and m_{ij} into Eq. (2), one finds that both equations yield

$$a_1^{(\pm)} = \tfrac{1}{2}(a_1 \pm i a_2),$$

$$a_2^{(\pm)} = \tfrac{1}{2}(a_2 \mp i a_1).$$

Hence, these two definitions are equivalent. A cursory inspection of these last two equations also shows that

$$\mathbf{a} = \mathbf{a}^{(+)} + \mathbf{a}^{(-)}.$$

Equation (2), however, has the advantage that one does not have to carry along factors of i in the subsequent derivations.

The following commutation relations will be useful in the sequel:

$$[m_{12}, a_1] = a_2, \tag{3a}$$

$$[m_{12}, a_2] = -a_1, \tag{3b}$$

$$[m_{12}, a_1^\dagger] = a_2^\dagger, \tag{3c}$$

$$[m_{12}, a_2^\dagger] = -a_1^\dagger. \tag{3d}$$

Using the commutation relations (1a) and (1b), we find that

$$[M, a_j^{(\pm)}] = \pm a_j^{(\pm)}.$$

It is now clear that $a_j^{(\pm)}$ define raising and lowering operators for M. Moreover, they satisfy, as already mentioned,

$$\mathbf{a} = \mathbf{a}^{(+)} + \mathbf{a}^{(-)}.$$

Let us now define the operators

$$a_1^{\dagger(+)} = (2M)^{-1}[a_1^\dagger(M+1) + a_2^\dagger(m_{21} + i)], \tag{4a}$$

$$a_2^{\dagger(+)} = (2M)^{-1}[a_2^\dagger(M+1) + a_1^\dagger(m_{12} - i)], \tag{4b}$$

$$a_1^{\dagger(-)} = (2M)^{-1}[a_1^\dagger(M-1) - a_2^\dagger(m_{21} - i)], \tag{4c}$$

$$a_2^{\dagger(-)} = (2M)^{-1}[a_2^\dagger(M-1) - a_1^\dagger(m_{12} + i)]. \tag{4d}$$

It can be readily shown by using Eqs. (3c) and (3d) that

$$[M, a_j^{\dagger(\pm)}] = \pm a_j^{\dagger(\pm)}. \tag{5}$$

We will verify $[M, a_1^{\dagger(-)}] = -a_1^{\dagger(-)}$.

In fact,

$$\begin{aligned} M a_1^{\dagger(-)} &= [M a_1^\dagger (M-1) - M a_2^\dagger (m_{21} - i)](2M)^{-1} \\ &= [(a_1^\dagger M + i a_2^\dagger)(M-1) \\ &\quad - (a_2^\dagger M - i a_1^\dagger)(m_{21} - i)](2M)^{-1} \\ &= \{[a_1^\dagger(M-1) - a_2^\dagger(m_{21} - i)]M \\ &\quad + i a_2^\dagger (M-1) + i a_1^\dagger (m_{21} - i)\}(2M)^{-1} \\ &= a_1^{\dagger(-)} M + [i a_1^\dagger (iM - i) + i a_2^\dagger (-i m_{21} - 1)](2M)^{-1} \\ &= a_1^{\dagger(-)} M - a_1^{\dagger(-)}. \end{aligned}$$ ■

The other equalitites in Eqs. (5.) can be similarly verified. Thus $a_j^{\dagger(+)}$ and $a_j^{\dagger(-)}$ $(j = 1, 2)$ are raising and lowering operators for the operator corresponding to M.

It can be readily shown that

$$[a_j^{\dagger(\pm)}]^\dagger = a_j^{(\mp)}. \tag{6}$$

Let us verify this for $[a_1^{\dagger(+)}]^\dagger = a_1^{(-)}$.
In fact,

$$\begin{aligned} [a_1^{(-)}]^\dagger &= (M a_1^\dagger - m_{21}^\dagger a_2^\dagger)/2M \\ &= (a_1^\dagger M + i a_2^\dagger - a_2^\dagger m_{12} + a_1^\dagger)/2M \\ &= (a_1^\dagger M + a_2^\dagger m_{21} + a_1^\dagger + i a_2^\dagger)/2M = a_1^{\dagger(+)}. \end{aligned}$$ ■

The other equalities in Eqs. (6) can be similarly verified.

We now define the Hermitian operator $K = \frac{1}{2}(N - M)$, i.e.,

$$N = 2K + M.$$

As in the three-dimensional case, the scalar operator N (and hence K) commutes with m_{12} and hence with M. Since

$$N\mathbf{a} = \mathbf{a}(N - 1),$$

it can be easily seen that the commutation relations between $\mathbf{a}^{(\pm)}$ and $\mathbf{a}^{\dagger(\pm)}$ on the one hand and K on the other hand are the same as in the three-dimensional situation, i.e.,

$$N\mathbf{a}^{(\pm)} = \mathbf{a}^{(\pm)}(N - 1),$$

$$[K, \mathbf{a}^{(-)}] = 0,$$

$$K\mathbf{a}^{(+)} = \mathbf{a}^{(+)}(K - 1),$$

$$[K, \mathbf{a}^{\dagger(+)}] = 0,$$

$$K\mathbf{a}^{\dagger(-)} = \mathbf{a}^{\dagger(-)}(K + 1).$$

We also have

$$N(\mathbf{a}\cdot\mathbf{a}) = (\mathbf{a}\cdot\mathbf{a})(N - 2),$$

where $(\mathbf{a}\cdot\mathbf{a})$ commutes with M.
Moreover,

$$K(\mathbf{a}\cdot\mathbf{a}) = (\mathbf{a}\cdot\mathbf{a})(K - 1)$$

and

$$K(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) = (\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})(K + 1).$$

As in the three-dimensional case, it can now be seen that $\mathbf{a}^{(+)}$ and $\mathbf{a}^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a})$ have the same shifting properties for N, K, and M.

From the definitions of $a_j^{(\pm)}$ and $a_j^{\dagger(\pm)}$, one can readily verify the following:

$$a_j^{(-)}(2M) = -a_j^{\dagger}(\mathbf{a}\cdot\mathbf{a}) + a_j(N + M), \tag{7}$$

$$a_j^{(+)}(2M) = a_j^{\dagger}(\mathbf{a}\cdot\mathbf{a}) - a_j(N - M), \tag{8}$$

$$a_1^{\dagger(-)}(2M) = a_1(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - a_1^{\dagger}(N - M + 3) + ia_2^{\dagger}, \tag{9}$$

$$a_2^{\dagger(-)}(2M) = a_2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - a_2^{\dagger}(N - M + 3) - ia_1^{\dagger}, \tag{10}$$

$$a_1^{\dagger(+)}(2M) = -a_1(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) + a_1^{\dagger}(N + M + 3) + ia_2^{\dagger}, \tag{11}$$

$$a_2^{\dagger(+)}(2M) = -a_2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) + a_2^{\dagger}(N + M + 3) - ia_1^{\dagger}. \tag{12}$$

Let us verify Eq. (10):

$$\begin{aligned}
a_2^{\dagger(-)}(2M) &= [a_2^{\dagger}(M - 1) - a_1^{\dagger}(m_{12} + i)] \\
&= a_2^{\dagger}M - a_2^{\dagger} - a_1^{\dagger}(a_1a_2^{\dagger} - a_2a_1^{\dagger}) - ia_1^{\dagger} \\
&= a_2^{\dagger}M - a_2^{\dagger} - a_1^{\dagger}a_1a_2^{\dagger} + a_1^{\dagger}a_2a_1^{\dagger} - ia_1^{\dagger} \\
&= a_2^{\dagger}M - a_2^{\dagger} - a_2^{\dagger}a_1^{\dagger}a_1 + a_2a_1^{\dagger}a_1^{\dagger} - ia_1^{\dagger} \\
&= a_2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - a_2a_2^{\dagger}a_2^{\dagger} - a_2^{\dagger}(a_1^{\dagger}a_1 - M + 1) - ia_1^{\dagger} \\
&= a_2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - a_2^{\dagger}(N - M + 3) - ia_1^{\dagger}.
\end{aligned}$$

■

The other equations can be similarly verified.

To enable us to derive relationships between $\mathbf{a}^{\dagger(-)}$ and $\mathbf{a}^{(-)}(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})$ and between $\mathbf{a}^{(+)}$ and $\mathbf{a}^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a})$, let us derive the relationship

$$(\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) = (N + M + 2)(N - M + 2), \tag{13}$$

from which follows the relationship

$$(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})(\mathbf{a}\cdot\mathbf{a}) = (N + M)(N - M). \tag{14}$$

We first note that $m_{12}m_{12} = -M^2$. Also, $m_{12}m_{12} = \frac{1}{2}(m_{ij}m_{ij}) = a_ia_j^{\dagger}a_ia_j^{\dagger} - a_ia_j^{\dagger}a_ja_i^{\dagger}$. Since $a_ia_j^{\dagger}a_ia_j^{\dagger} = (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - N - 2$ and $a_ia_j^{\dagger}a_ja_i^{\dagger} = N^2 + 3N + 2$, we obtain

$$(\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - N - 2 - N^2 - 3N - 2 = -M^2,$$

i.e., $(\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) = (N + M + 2)(N - M + 2)$. ■

The following relationships are found between $\mathbf{a}^{\dagger(-)}$ and $\mathbf{a}^{(-)}(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})$:

$$a_1^{\dagger(-)} = a_1^{(-)}(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})(2K + 2M + 2)^{-1} - (2M)^{-1}(a_1^{\dagger} - ia_2^{\dagger}), \tag{15}$$

$$a_2^{\dagger(-)} = a_2^{(-)}(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})(2K + 2M + 2)^{-1} - (2M)^{-1}(a_2^{\dagger} + ia_1^{\dagger}). \tag{16}$$

Similarly,

$$a_1^{(+)} = [a_1^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a}) - (2M)^{-1}(a_1^{\dagger} + ia_2^{\dagger})(\mathbf{a}\cdot\mathbf{a})] \times (N + M)^{-1}, \tag{17}$$

$$a_2^{(+)} = [a_2^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a}) - (2M)^{-1}(a_2^{\dagger} - ia_1^{\dagger})(\mathbf{a}\cdot\mathbf{a})] \times (N + M)^{-1}. \tag{18}$$

Let us note that Eqs. (17) and (18) are not defined on vectors on which N and M vanish. On such vectors, one may define $\mathbf{a}^{(+)} = 0$.

We will verify Eq. (18) as an example of these relationships. Let us start by multiplying Eq. (12) by $(\mathbf{a}\cdot\mathbf{a})$ on the right, obtaining

$$\begin{aligned}
a_2^{\dagger(+)}&(\mathbf{a}\cdot\mathbf{a})(2M) \\
&= -a_2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})(\mathbf{a}\cdot\mathbf{a}) + a_2^{\dagger}(N + M + 3)(\mathbf{a}\cdot\mathbf{a}) - ia_1^{\dagger}(\mathbf{a}\cdot\mathbf{a}) \\
&= -a_2(N + M)(N - M) + a_2^{\dagger}(\mathbf{a}\cdot\mathbf{a}) \\
&\quad \times (N + M + 1) - ia_1^{\dagger}(\mathbf{a}\cdot\mathbf{a})
\end{aligned}$$

[by Eq. (14) and because $(\mathbf{a}\cdot\mathbf{a})$ is a lowering operator for K]

$$\begin{aligned}
&= [a_2^{\dagger}(\mathbf{a}\cdot\mathbf{a}) - a_2(N - M)](N + M) + (a_2^{\dagger} - ia_1^{\dagger})(\mathbf{a}\cdot\mathbf{a}) \\
&= a_2^{(+)}(2M)(N + M) + (a_2^{\dagger} - ia_1^{\dagger})(\mathbf{a}\cdot\mathbf{a})
\end{aligned}$$

[by using Eq. (8)],

$$\text{i.e.,}\quad a_2^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a}) = a_2^{(+)}(N + M) + (2M)^{-1}(a_2^{\dagger} - ia_1^{\dagger})(\mathbf{a}\cdot\mathbf{a}),$$

$$\text{i.e.,}\quad a_2^{(+)} = [a_2^{\dagger(+)}(\mathbf{a}\cdot\mathbf{a}) - (2M)^{-1}(a_2^{\dagger} - ia_1^{\dagger})(\mathbf{a}\cdot\mathbf{a})] \times (N + M)^{-1}.$$

■

The other equations can be similarly verified.

As in the three-dimensional case, the fundamental operators are $\mathbf{a}^{\dagger(+)}$ and its conjugate $\mathbf{a}^{(-)}$. These are raising and lowering operators for M but commute with K. $(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})$ and $(\mathbf{a}\cdot\mathbf{a})$ are also fundamental since they are raising and lowering operators for K but commute with M. Let us also regard as equally fundamental the operators λ, ν, and their conjugates $\lambda^{\dagger}$ and $\nu^{\dagger}$ defined by

$$\lambda = f(K, M + 1)\mathbf{a}^{(-)} = \mathbf{a}^{(-)}f(K, M),$$

$$\lambda^{\dagger} = f(K, M)\mathbf{a}^{\dagger(+)} = \mathbf{a}^{\dagger(+)}f(K, M + 1),$$

$$\nu = g(K+1,M)(\mathbf{a}\cdot\mathbf{a}) = (\mathbf{a}\cdot\mathbf{a})g(K,M),$$
$$\nu^\dagger = g(K,M)(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) = (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)g(K+1,M),$$

where f and g are Hermitian operator functions. These operators have the same shifting properties for K and M as do $\mathbf{a}^{(-)}$, $\mathbf{a}^{\dagger(+)}$, $(\mathbf{a}\cdot\mathbf{a})$, and $(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)$, respectively.

The following commutation relations are the same as in the three-dimensional case:

$$M\lambda = \lambda(M-1), \quad M\lambda^\dagger = \lambda^\dagger(M+1), \tag{19}$$
$$[M,\nu] = 0 = [M,\nu^\dagger], \tag{20}$$
$$K\nu = \nu(K-1), \quad K\nu^\dagger = \nu^\dagger(K+1), \tag{21}$$
$$[K,\lambda] = 0 = [K,\lambda^\dagger]. \tag{22}$$

It can be readily shown that

$$\begin{aligned}\lambda_i\lambda_j =& f(K,M+1)f(K,M+2)\\ &\times(2M+2)^{-1}(2M+4)^{-1}a_i^{(-)}(2M)a_j^{(-)}(2M)\\ =& f(K,M+1)f(K,M+2)(2M+2)^{-1}\\ &\times(2M+4)^{-1}[a_i^\dagger a_j^\dagger(\mathbf{a}\cdot\mathbf{a})^2\\ &+ a_ia_j(2K+2M)(2K+2M-2)\\ &- (a_i^\dagger a_i + a_ia_j^\dagger)(\mathbf{a}\cdot\mathbf{a})(2K+2M-2)].\end{aligned} \tag{23}$$

Since the rhs of this equation is symmetric in i and j

$$[\lambda_i,\lambda_j] = 0.$$

By Hermitian conjugation

$$[\lambda_i^\dagger,\lambda_j^\dagger] = 0.$$

We now proceed to evaluate $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$.

From Eq. (23) we have

$$\begin{aligned}(2M+2)(2M+4)(\mathbf{a}^{(-)}\cdot\mathbf{a}^{(-)}) =& (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a})^2\\ &+ (\mathbf{a}\cdot\mathbf{a})(2K+2M)(2K+2M-2)\\ &- (a_i^\dagger a_i + a_ia_i^\dagger)(\mathbf{a}\cdot\mathbf{a})(2K+2M-2)\\ =& [(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(\mathbf{a}\cdot\mathbf{a}) + (2K+2M+2)(2K+2M)\\ &- (2N+2)(2K+2M)](\mathbf{a}\cdot\mathbf{a})\\ =& [(N+M)(N-M) + (N+M+2)(N+M)\\ &- (2N+2)(N+M)](\mathbf{a}\cdot\mathbf{a}) = 0.\end{aligned}$$

Thus

$$\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = 0.$$

By taking the Hermitian conjugate of Eq. (24) we obtain

$$\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}^\dagger = 0 \tag{24}$$

It is trivial to verify that

$$i[\lambda_1,m_{12}] = \lambda_1,$$
$$i[\lambda_1^\dagger,m_{12}] = -\lambda_1^\dagger,$$

both of which are different from their three-dimensional counterparts.

To compute the remaining commutation relations, let us choose

$$f(K,M) = [(M-1)/(K+M)]^{1/2}, \tag{25}$$
$$g(K,M) = (4K+4M)^{-1/2}. \tag{26}$$

On vectors on which K and M vanish, one may define

$$\lambda = 0, \quad \nu = 0.$$

Then we find that

$$[\nu,\nu^\dagger] = 1 \text{ and } K = \nu^\dagger\nu.$$

Similarly,

$$[\lambda_i,\nu] = 0 = [\lambda_i^\dagger,\nu^\dagger].$$

We now show that

$$[\lambda_i,\nu^\dagger] = 0. \tag{27}$$

In fact,

$$\lambda_i\nu^\dagger = a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)f(K+1,M)g(K+1,M).$$

Also

$$\begin{aligned}\nu^\dagger\lambda_i =& (\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)a_i^{(-)}g(K+1,M-1)f(K,M)\\ =& [a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2a_n^\dagger\\ &\times(M\delta_{ni} - m_{ni})(2M)^{-1}]g(K+1,M-1)f(K,M).\end{aligned}$$

By using Eqs. (1b) and (2) and the commutation of $(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)$ with M and m_{ni}, we obtain

$$\begin{aligned}\nu^\dagger\lambda_1 =& \{a_1^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2[a_1^{\dagger(-)} + (2M)^{-1}(a_1^\dagger - ia_2^\dagger)]\}\\ &\times g(K+1,M-1)f(K,M),\\ \nu^\dagger\lambda_2 =& \{a_2^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2[a_2^{\dagger(-)} + (2M)^{-1}(a_2^\dagger + ia_1^\dagger)]\}\\ &\times g(K+1,M-1)f(K,M)\end{aligned}$$

[by using Eqs. (4c) and (4d)], i.e.,

$$\begin{aligned}\nu^\dagger\lambda_1 =& (a_1^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2a_1^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2M+2)^{-1})\\ &\times g(K+1,M-1)f(K,M),\\ \nu^\dagger\lambda_2 =& [a_2^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2a_2^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2M+2)^{-1}]\\ &\times g(K+1,M-1)f(K,M),\end{aligned}$$

[by using Eqs. (15) and (16)], i.e.,

$$\begin{aligned}\nu^\dagger\lambda_i =& [a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) - 2a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2M+2)^{-1}]\\ &\times g(K+1,M-1)f(K,M).\end{aligned}$$

Thus

$$\begin{aligned}\lambda_i\nu^\dagger - \nu^\dagger\lambda_i =& a_i^{(-)}(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger)(2K+2M+2)^{-1}[(2K+2M+2)\\ &\times f(K+1,M)g(K+1,M) - (2K+2M)\\ &\times g(K+1,M-1)f(K,M)] = 0\end{aligned}$$

[by substituting Eqs. (25) and (26)]. ■

By taking the Hermitian conjugate of Eq. (27), we obtain

$$[\lambda_i^\dagger,\nu] = 0.$$

By using Eqs. (11), (2), (12), (1a), and (1b), we obtain

$$\begin{aligned}\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} \equiv& \lambda_1^\dagger\lambda_1 + \lambda_2^\dagger\lambda_2 = a_1^{\dagger(+)}(2M)a_1^{(-)}f^2(K,M)(2M-2)^{-1}\\ &+ a_2^{\dagger(+)}(2M)a_2^{(-)}f^2(K,M)(2M-2)^{-1}\\ =& [-(\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^\dagger\cdot\mathbf{a}^\dagger) + (2K+M)\\ &\times(2K+2M+3) + 4 + ia_2^\dagger a_1(2K+2M)\\ &- i(a_1^\dagger a_2)(2K+2M) + (2K+M)]\\ &\times f^2(K,M)[2(2M-2)]^{-1}\\ =& [-(2K+2M+2)(2K+2)\\ &+ (2K+M)(2K+2M+3) + 4\\ &+ M(2K+2M) + (2K+M)]\\ &\times f^2(K,M)[2(2M-2)]^{-1}\end{aligned}$$

[by using Eq. (13) and the definition of m_{12}]

$= M$

by substituting Eq. (25) for f. Thus

$$\boldsymbol{\lambda}^{\dagger}\cdot\boldsymbol{\lambda} = M.$$

Since $N = 2K + M$, $K = \nu^{\dagger}\nu$, and $M = \boldsymbol{\lambda}^{\dagger}\cdot\boldsymbol{\lambda}$, the Hamiltonian of the isotropic two-dimensional harmonic oscillator in an angular momentum basis can be written in the form

$$H = \hbar\omega(N + 1) = \hbar\omega(2\nu^{\dagger}\nu + \boldsymbol{\lambda}^{\dagger}\cdot\boldsymbol{\lambda} + 1).$$

Correspondingly, the eigenvalues of this Hamiltonian operator are of the form

$$E_{km} = \hbar\omega(2K + |m| + 1),$$

where $k = 0, 1, 2, \cdots$, and $m = 0, \pm 1, \pm 2, \cdots$.

We will finally show that

$$il_{ij} = \lambda_i^{\dagger}\lambda_j - \lambda_j^{\dagger}\lambda_i.$$

Specifically, let us choose $i = 1, j = 2$,

$$\begin{aligned}\lambda_1^{\dagger}\lambda_2 &= f(K,M)a_1^{\dagger(+)}a_2^{(-)}f(K,M)\\ &= a_1^{\dagger(+)}(2M)a_2^{(-)}f^2(K,M)(2M-2)^{-1},\end{aligned}$$

[by Eqs. (9) and (2)]

$$\begin{aligned}&= [\,-a_1a_2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) + a_2^{\dagger}a_1 - i(a_1)^2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) + ia_1^{\dagger}a_1\\ &\quad\times(2K + 2M + 3) + 2i + a_1^{\dagger}a_2\\ &\quad\times(2K + 2M + 1) + ia_2^{\dagger}a_2]\\ &\quad\times f^2(K,M)[4(M-1)]^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_2^{\dagger}\lambda_1 &= [\,-a_2a_1(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) + a_2a_1^{\dagger} + i(a_2)^2(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger}) - ia_2^{\dagger}a_2\\ &\quad\times(2K + 2M + 3) - 2i + a_2^{\dagger}a_1(2K + 2M + 1)\\ &\quad - ia_1^{\dagger}a_1]f^2(K,M)[4(M-1)]^{-1}\end{aligned}$$

(by substituting $N = 2K + M = a_1^{\dagger}a_1 + a_2^{\dagger}a_2$)

$$\begin{aligned}\therefore \lambda_1^{\dagger}\lambda_2 - \lambda_2^{\dagger}\lambda_1 &= \{\,-i(\mathbf{a}\cdot\mathbf{a})(\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})\\ &\quad + i(2K + M)(2K + 2M + 3) + 4i\\ &\quad + i(2K + M) + (a_1a_2^{\dagger} - a_2a_1^{\dagger})\\ &\quad + (a_2a_1^{\dagger} - a_1a_2^{\dagger})(2K + 2M + 1)\}\\ &\quad\times(M-1)/[4(M-1)(K+M)]\end{aligned}$$

[by Eq. (13) and the definition of m_{ij}]

$$= iM = il_{12}.$$ ■

The commutation relation

$$(2\boldsymbol{\lambda}^{\dagger}\cdot\boldsymbol{\lambda} + 1)[\lambda_i, \lambda_j^{\dagger}] = (2\boldsymbol{\lambda}^{\dagger}\cdot\boldsymbol{\lambda} + 1)\delta_{ij} - 2\lambda_i^{\dagger}\lambda_j,$$

which holds in the three-dimensional case, also holds in the two-dimensional situation.

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[1] A. J. Bracken and H. I. Leemon, J. Math. Phys. **21**, 2170–2181 (1980).
[2] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961).
[3] A. J. Bracken and H. S. Green, J. Math. Phys. **12**, 2099 (1971).

A time-dependent Schrödinger equation

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We reduce the solution of the Schrödinger equation with the potential

$$U(r,t)=\alpha_1(t)\,\mathbf{r}^2+\alpha_2(t)\,x+\alpha_3(t)\,y+\alpha_4(t)\,z+\alpha_5(t)$$

to the solution of the Schrödinger equation for a free particle. The $\alpha_i(t)$ are arbitrary functions of time. A generalization of this is also considered.

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I. INTRODUCTION

The solution of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}+\alpha_1(t)\,x^2+\alpha_2(t)\,x+\alpha_3(t)$$

has been treated in the literature (see Ref. 1 and the references given there). A generalization to three dimensions was considered in Ref. 2. In this paper, we reduce the solution of the Schrödinger equation for the potential

$$U=\alpha_1\mathbf{r}^2+\alpha_2x+\alpha_3y+\alpha_4z+\alpha_5,$$

where the α_i are arbitrary functions of the time to the solution of the Schrödinger equation for a free particle. We also consider the generalization where

$$\frac{1}{b^2}K\left(\frac{\mathbf{r}-\mathbf{f}(t)}{b}\right)$$

is added to the above potential. The resulting Schrödinger equation is reduced to a Schrödinger equation with the potential $K(\mathbf{r})$, where K is an arbitrary function.

II. TRANSFORMATIONS

Consider the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\nabla^2\psi+U(\mathbf{r})\,\psi$$

and the transformations

$$\psi=\psi(\mathbf{r},t),$$
$$t'=t'(\mathbf{r},t),$$
$$\mathbf{r}'=\mathbf{r}'(\mathbf{r},t),$$
$$\psi(r,t)=A(\mathbf{r}',t')\,e^{iS(\mathbf{r}',t')}\,\psi'(\mathbf{r}',t');$$

The resulting equation now has the form

$$\begin{aligned}ig_1\frac{\partial\psi'}{\partial t'}&+g_2\frac{\partial^2\psi'}{\partial t'^2}+g_3\frac{\partial^2\psi'}{\partial x'^2}+g_4\frac{\partial^2\psi'}{\partial y'^2}+g_5\frac{\partial^2\psi'}{\partial z'^2}\\&+g_6\frac{\partial\psi'}{\partial x'}+g_7\frac{\partial\psi'}{\partial y'}+g_8\frac{\partial\psi'}{\partial z'}+g_9\frac{\partial^2\psi'}{\partial x'\partial t'}\\&+g_{10}\frac{\partial^2\psi'}{\partial y'\partial t'}+g_{11}\frac{\partial^2\psi'}{\partial z'\partial t'}g_{12}\frac{\partial^2\psi'}{\partial x'\partial y'}\\&+g_{13}\frac{\partial^2\psi'}{\partial x'\partial z'}+g_{14}\frac{\partial^2\psi'}{\partial y'\partial z'}+g_{15}\psi'=0.\end{aligned}$$

We would like the form of this equation for ψ' to be that of a Schrödinger equation.

For this, we must have

$$g_1=g_3=g_4=g_5,$$
$$g_2=0,$$
$$g_6=g_7=g_8=0,$$
$$g_9=g_{10}=g_{11}=0,$$
$$g_{12}=g_{13}=g_{14}=0,$$
$$\mathrm{Im}(g_{15}/g_1)=0.$$

The potential of this Schrödinger equation for ψ' is then $-\mathrm{Re}(g_{15}/g_1)$.

Now

$$g_2=A\,e^{iS}\left[\left(\frac{\partial t'}{\partial x}\right)^2+\left(\frac{\partial t'}{\partial y}\right)^2+\left(\frac{\partial t'}{\partial z}\right)^2\right]=0,$$

which implies that

$$t'=T(t), \tag{2.1}$$

$$\begin{aligned}g_{12}&=2\left[\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x}+\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y}+\frac{\partial x'}{\partial z}\frac{\partial y'}{\partial z}\right]\\&\times A\,e^{iS}=0,\\g_{13}&=2\left[\frac{\partial z'}{\partial x}\frac{\partial x'}{\partial x}+\frac{\partial z'}{\partial y}\frac{\partial x'}{\partial y}+\frac{\partial z'}{\partial z}\frac{\partial x'}{\partial z}\right]\\&\times A\,e^{iS}=0,\\g_{14}&=2\left[\frac{\partial z'}{\partial x}\frac{\partial y'}{\partial x}+\frac{\partial z'}{\partial y}\frac{\partial y'}{\partial y}+\frac{\partial z'}{\partial z}\frac{\partial y'}{\partial z}\right]\\&\times A\,e^{iS}=0,\end{aligned} \tag{2.2}$$

$$\begin{aligned}g_3&=\left[\left(\frac{\partial x'}{\partial x}\right)^2+\left(\frac{\partial x'}{\partial y}\right)^2+\left(\frac{\partial x'}{\partial z}\right)^2\right]A\,e^{iS},\\g_4&=\left[\left(\frac{\partial y'}{\partial x}\right)^2+\left(\frac{\partial y'}{\partial y}\right)^2+\left(\frac{\partial y'}{\partial z}\right)^2\right]A\,e^{iS},\\g_5&=\left[\left(\frac{\partial z'}{\partial x}\right)^2+\left(\frac{\partial z'}{\partial y}\right)^2+\left(\frac{\partial z'}{\partial z}\right)^2\right]A\,e^{iS},\end{aligned} \tag{2.3}$$

$$g_1=\frac{\partial t'}{\partial t}A\,e^{iS}.$$

Since $g_1=g_3=g_4=g_5$, we now have

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$$\begin{pmatrix} \frac{\partial x'}{\partial x}, & \frac{\partial x'}{\partial y}, & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x}, & \frac{\partial y'}{\partial y}, & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x}, & \frac{\partial z'}{\partial y}, & \frac{\partial z'}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x'}{\partial x}, & \frac{\partial y'}{\partial x}, & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y}, & \frac{\partial y'}{\partial y}, & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z}, & \frac{\partial y'}{\partial z}, & \frac{\partial z'}{\partial z} \end{pmatrix}$$

$$= \frac{\partial t'}{\partial t} \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix}. \tag{2.4}$$

Hence the matrix

$$\frac{1}{\sqrt{(\partial t'/\partial t)}} \frac{\partial(\ ')}{\partial(\ )}$$

is orthogonal, i.e.,

$$\frac{\partial(\ ')}{\partial(\ )} = b(t)\, R, \tag{2.5}$$

where R is orthogonal. Therefore,

$$\frac{\partial t'}{\partial t} = b^2(t) \tag{2.6}$$

and

$$t' = \int^t b^2(\tau)\, d\tau + \text{const.}$$

Now

$$\begin{aligned} g_6 = iA\, e^{iS} \frac{\partial x'}{\partial t} &+ 2\left[\left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial x'}{\partial y}\right)^2 + \left(\frac{\partial x'}{\partial z}\right)^2\right] \frac{\partial}{\partial x'}(A\, e^{iS}) \\ &+ 2\left[\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} + \frac{\partial x'}{\partial z}\frac{\partial y'}{\partial z}\right] \frac{\partial}{\partial y'}(A\, e^{iS}) \\ &+ 2\left[\frac{\partial x'}{\partial x}\frac{\partial z'}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial z'}{\partial y} + \frac{\partial x'}{\partial z}\frac{\partial z'}{\partial z}\right] \frac{\partial}{\partial z'}(A\, e^{iS}), \end{aligned}$$

and similarly for g_7 and g_8.

Because of (2.4), we therefore have

$$g_6 = iA\, e^{iS} \frac{\partial x'}{\partial t} + 2b^2 \frac{\partial}{\partial x'}(A\, e^{iS}),$$

$$g_7 = iA\, e^{iS} \frac{\partial y'}{\partial t} + 2b^2 \frac{\partial}{\partial y'}(A\, e^{iS}),$$

$$g_8 = iA\, e^{iS} \frac{\partial z'}{\partial t} + 2b^2 \frac{\partial}{\partial z'}(A\, e^{iS}).$$

For these to vanish, we have

$$\text{(a)} \quad \frac{\partial A}{\partial x'} = \frac{\partial A}{\partial y'} = \frac{\partial A}{\partial z'} = 0$$

and, therefore,

$$A = A(t) \tag{2.7}$$

and

$$\text{(b)} \quad \begin{aligned} \frac{\partial x'}{\partial t} + 2b^2 \frac{\partial S}{\partial x'} &= 0, \\ \frac{\partial y'}{\partial t} + 2b^2 \frac{\partial S}{\partial y'} &= 0, \\ \frac{\partial z'}{\partial t} + 2b^2 \frac{\partial S}{\partial z'} &= 0. \end{aligned} \tag{2.8}$$

We now restrict ourselves to linear transformations, that is, R is a constant matrix. Then it follows from (2.5) that

$$\mathbf{r}' = b(\tau(t'))\, R\mathbf{r} + \mathbf{f}(t'), \tag{2.9}$$

where $t = \tau(t')$ and where $\mathbf{f}(t')$ is an arbitrary time-dependent vector. Now Eqs. (2.8),

$$\frac{\partial \mathbf{r}'}{\partial t} + 2b^2 \nabla' S = 0,$$

may be solved for S:

$$\dot{b} R\mathbf{r} + \dot{\mathbf{f}} + 2b^2 \nabla' S = 0,$$

where the dot $(\cdot)$ denotes differentiation with respect to t, or

$$b' R\mathbf{r} + \mathbf{f}' + 2\nabla' S = 0,$$

where the prime $(')$ denotes differentiation with respect to t'.

Elimination of $R\mathbf{r}$ now results in

$$\frac{\partial S}{\partial x_i'} = -\frac{1}{2}\left[\frac{b'}{b} x_i' + \left\{f_i' - \frac{b'}{b} f_i\right\}\right].$$

Hence,

$$\begin{aligned} S = &-\frac{1}{4}\frac{b'}{b}\mathbf{r}^2 - \frac{1}{2}\left[f_1' - \frac{b'}{b} f_1\right] x_1' \\ &- \frac{1}{2}\left[f_2' - \frac{b'}{b} f_2\right] x_2' \\ &- \frac{1}{2}\left[f_3' - \frac{b'}{b} f_3\right] x_3' + F(t'), \end{aligned} \tag{2.10}$$

where F is an arbitrary function of t'.

Now

$$\begin{aligned} \frac{g_{15}}{g_1} &= \frac{g_{15}}{b^2 A\, e^{iS}} \\ &= \frac{i}{A}\frac{\partial A}{\partial t'} - \frac{\partial S}{\partial t'} + 2\left(\frac{\partial S}{\partial x'}\right)^2 + 2\left(\frac{\partial S}{\partial y'}\right)^2 + 2\left(\frac{\partial S}{\partial z'}\right)^2 \\ &\quad + i\frac{\partial^2 S}{\partial x'^2} + i\frac{\partial^2 S}{\partial y'^2} + i\frac{\partial^2 S}{\partial z'^2} - \frac{U(\mathbf{r})}{b^2}. \end{aligned}$$

For $\operatorname{Im}(g_{15}/g_1)$ to vanish,

$$\frac{1}{A}\frac{\partial A}{\partial t'} = -\left[\frac{\partial^2 S}{\partial x'^2} + \frac{\partial^2 S}{\partial y'^2} + \frac{\partial^2 S}{\partial z'^2}\right] = +\frac{3}{2}\frac{b'}{b}$$

or

$$A = kb^{3/2}. \tag{2.11}$$

Now

$$\begin{aligned} \operatorname{Re}\left(\frac{g_{15}}{g_1}\right) = &-\frac{\partial S}{\partial t'} + 2\left[\left(\frac{\partial S}{\partial x'}\right)^2 + \left(\frac{\partial S}{\partial y'}\right)^2 + \left(\frac{\partial S}{\partial z'}\right)^2\right] \\ &- \frac{U(r)}{b^2} = -U'(\mathbf{r}', t'), \end{aligned} \tag{2.12}$$

where U' is the potential in the Schrödinger equation for $\psi'(\mathbf{r}', t')$.

By using (2.10), we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{g_{15}}{g_1}\right) = & -\frac{1}{4}\left\{\left(\frac{b'}{b}\right)' - \left(\frac{b'}{b}\right)^2\right\}(x^2+y^2+z^2)\\ & +\frac{1}{2}\left\{\left[f_1' - \left(\frac{b'}{b}\right)f_1\right]' + \frac{b'}{b}\left[f_1' - \left(\frac{b'}{b}\right)f_1\right]\right\}x'\\ & +\frac{1}{2}\left\{\left[f_2' - \left(\frac{b'}{b}\right)f_2\right]' + \frac{b'}{b}\left[f_2' - \left(\frac{b'}{b}\right)f_2\right]\right\}y'\\ & +\frac{1}{2}\left\{\left[f_3' - \left(\frac{b'}{b}\right)f_3\right]' + \frac{b'}{b}\left[f_3' - \left(\frac{b'}{b}\right)f_3\right]\right\}z'\\ & -\left\{F' - \frac{1}{4}\left[f_1' - \left(\frac{b'}{b}\right)f_1\right]^2 - \frac{1}{4}\left[f_2' - \left(\frac{b'}{b}\right)f_2\right]^2\right.\\ & \left.-\frac{1}{4}\left[f_3' - \left(\frac{b'}{b}\right)f_3\right]^2\right\} - \frac{U(\mathbf{r})}{b^2}.\end{aligned}$$

Hence the transformed potential is

$$\begin{aligned}U'(\mathbf{r}', t') = & \alpha_1(t')\,\mathbf{r}'^2 + \alpha_2(t')\,x' + \alpha_3(t')\,y'\\ & + \alpha_4(t')\,z' + \alpha_5(t') + b^{-2}U(\tilde{R}(\mathbf{r}' - \mathbf{f}(t'))/b).\end{aligned} \tag{2.13}$$

If now $\alpha_i(t')$, $i = 1,\ldots,5$ are considered to be arbitrarily given, the functions b, $\mathbf{f}$, and F may be obtained by solving the following equations:

$$\alpha_1(t') = \frac{1}{4}\left\{\left(\frac{b'}{b}\right)' - \left(\frac{b'}{b}\right)^2\right\}, \tag{2.14}$$

$$\alpha_2(t') = -\frac{1}{2}\left\{\left[f_1' - \left(\frac{b'}{b}\right)f_1\right]' + \frac{b'}{b}\left[f_1' - \left(\frac{b'}{b}\right)f_1\right]\right\}, \tag{2.15}$$

$$\alpha_3(t') = -\frac{1}{2}\left\{\left[f_2' - \left(\frac{b'}{b}\right)f_2\right]' + \frac{b'}{b}\left[f_2' - \left(\frac{b'}{b}\right)f_2\right]\right\}, \tag{2.16}$$

$$\alpha_4(t') = -\frac{1}{2}\left\{\left[f_3' - \left(\frac{b'}{b}\right)f_3\right]' + \frac{b'}{b}\left[f_3' - \left(\frac{b'}{b}\right)f_3\right]\right\}, \tag{2.17}$$

$$\begin{aligned}\alpha_5(t') = & \left\{F' - \frac{1}{4}\left[f_1' - \left(\frac{b'}{b}\right)f_1\right]^2 - \frac{1}{4}\left[f_2' - \left(\frac{b'}{b}\right)f_2\right]^2\right.\\ & \left.-\frac{1}{4}\left[f_3' - \left(\frac{b'}{b}\right)f_3\right]^2\right\}.\end{aligned} \tag{2.18}$$

Equation (2.14) gives b in terms of α_1, (2.15) gives f_1 in terms of α_2, (2.16) gives f_2 in terms of α_3, (2.17) gives f_3 in terms of α_4, and (2.18) gives F in terms of α_5.

III. CONCLUSION

We have reduced the solution of the Schrödinger equation with the potential (2.13) to the solution of a Schrödinger equation with the potential $U(\mathbf{r})$ plus the solution of a number of ordinary differential equations (2.14)–(2.18). This may be extended as in Ref. 2 to potentials cubic in the coordinates. Then, however, the time-dependent coefficients depend on one another.

The solution of (2.15)–(2.17) is given by

$$\begin{aligned}f_i(t') = b(t')\left[\int^{t'}\frac{1}{b^2(\tau)}\left\{2\int^{\tau} b(\tau')\,\alpha_{i+1}(\tau')\,d\tau'\right.\right.\\ \left.\left.+ k_i\right\}d\tau + k_i'\right].\end{aligned}$$

The solution of (2.18) is immediate and there remains only (2.14) to be solved. For $\alpha_1 = 0$, we can solve (2.14) analytically:

$$b(t) = k_4'/(t' + k_4),$$

where the k_i and k_i' are arbitrary constants. For $\alpha_1(t') \neq 0$, (2.14) is the generalized Ricatti equation discussed in Ref. 3 with b'/b as the dependent variable.

APPENDIX

We wish to illustrate the kind of transformations the above problem leads to for the special example of the harmonic oscillator. For this, we set $\alpha_1(t) = \sigma/4$, a constant, and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. Then it follows from (2.14) that

$$b'/b = -\sigma^{1/2}\cot\sigma^{1/2}(t' + k_4)$$

and, hence,

$$b = k_4'\sin^{-1}\sigma^{1/2}(t' + k_4).$$

Now it follows from (2.6) that

$$t = \int^{t'} k_4'^{-2}\,|\sin\sigma^{1/2}(\tau + k_4)|^2 + k_5,$$

and from (2.15)–(2.17), it follows that

$$f_i = k_i\,b\int^{t'}\frac{d\tau''}{b^2(\tau(\tau'))} + k_i', \quad i = 1,2,3,$$

where the k_i and k_i' are arbitrary constants. It follows that for certain values of t such that

$$t' = -k_4 + n\pi/\sigma^{1/2}, \quad n = 0,1,2,\cdots,$$

the scaling factor b becomes infinite and hence all points $\mathbf{r}$ are mapped to points $\mathbf{r}'$ where $\mathbf{r}'$ is infinite. Since this transformation reduces the harmonic oscillation to a free particle, one might expect this divergence to occur at times analogous to the classical times of the turning points of the particle.

[1] J. A. Ray, Phys. Rev. A **26**, 729 (1982).
[2] H. U. Kranold (to be published).
[3] H. T. Dais, *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962).

Lorentz covariance of an extended object in the tree approximation. II. Nonspherical object in 3 + 1 dimensions

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This is the second in the series of the papers in which we investigate the Lorentz covariance of the extended object. In this paper we examine the covariance of the deformed object in 3 + 1 dimensions in the tree approximation. We construct the solution of the Euler equation, which is Lorentz covariant. In such a covariant solution, the variables associated with the rotational and the translational zero modes appear as classical quantum mechanical operators. Consequently the covariant solution has an intrinsic spin, in addition to the intrinsic quantum mechanical momenta. Then, at the end of this work we will show that such a covariant solution can be obtained also by quantizing a classical solution of the Euler equation, having extra variables signifying the center and the orientation of the deformed object. In the tree approximation, the energy–momentum and the relativistic angular momentum of the extended object ψ become pure classical quantum mechanical operators, having been integrated over the space. Then it is proven that such four-momenta and angular momentum operators form a classical quantum mechanics presented in a relativistic manner. The center of mass of the extended object, often called collective coordinate, is shown to be made of these four-momentum and angular momentum. This center of mass and the four-momentum operators form a quantum mechanics presented in the conventional form.

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I. INTRODUCTION

When the vacuum value ϕ of the Heisenberg field ψ, namely, the order parameter $\phi = \langle\psi\rangle$, is a local function of x_μ, the function ϕ obeys a classical Euler equation of motion in the tree approximation.[1] Further the collective coordinates introduced in order to indicate the center of mass, were found to be classical quantum mechanical operators.[2] Then such a Heisenberg field ψ, nowadays called an extended object, contains a classical field and classical quantum mechanical operators as well as the radiation fields (second quantized fields). In the present paper we particularly pay attention to the quantum mechanical operators appearing in the deformed extended object ψ in 3 + 1 dimensions.

Since its creation in 1974, the theory of the extended object has been actively investigated in various models, either by the conventional perturbation method or by the path integral formalism. In these studies, however, there has been always an underlying assumption that such a Heisenberg field ψ satisfies the canonical commutation relations. Nevertheless, the canonical commutation relationships of ψ have been studied rather little. In fact, when we had begun investigating it, we found that the $\psi(x_\mu)$ as well as $\psi(x_\mu + a_\mu)$ commonly used, do not satisfy the relativistic form of canonical commutation relations.[3–5] In order for the ψ to satisfy the canonical commutation relations, the x_μ (or $x_\mu + a_\mu$) have to be replaced by X_μ, where X_μ are the coordinates of the moving particle with quantum mechanical velocity and the center of mass operator [see (2.10), (8.5), (8.6) and (8.8) of I]. In the extreme nonrelativistic limit, namely, when we ignore the quantum velocity, X_μ are reduced to $x_\mu + a_\mu$.

Now, if the field ψ satisfies the canonical commutation relations, then it fulfils the conservation relations $[\partial_\mu + iP_\mu, \psi] = 0$ and $[x_\mu\partial_\nu - x_\nu\partial_\mu + im_{\mu\nu}, \psi] = 0$. Here P_μ and $m_{\mu\nu}$ are the linear and angular momenta. In other words, the Heisenberg field must be Poincaré covariant. Since a few years ago two groups of physicists, one at Alberta (Canada) and the other at Strasbourg (France), have been investigating the Lorentz covariance of ψ by making use of these conservation relations. In the previous paper I, we have determined the covariant form of Heisenberg fields for a spherical object in 3 + 1 dimensions and a deformed object in 2 + 1 dimensions. In this second part II, we will deal with a deformed object in 3 + 1 dimensions.

Compared to the case of spherical case, there occur in the deformed object a new kind of zero mode, namely the rotational zero modes $(x_i\partial_j - x_j\partial_i)\psi$. In 2 + 1 dimensions, there occurs only one rotational mode as seen in I. In the case of 3 + 1 dimensions, however, there appear three rotational zero modes, in addition to the translational zero modes. Then the field ψ has to include three more variables corresponding to these rotational zero modes, in addition to the collective coordinates (center of mass).

Since the rotational operator $(x\partial_y - y\partial_x)$'s are nonabelian, these new variables do not commute among themselves. The presence of such variables in the Heisenberg field ψ makes our task very complicated. Our aim is to determine explicitly the Heisenberg field ψ, having these new variables and satisfying the conservation relations mentioned previously. In order to do this we divide the procedure into two steps. As a first step we examine the vacuum value of ψ in the tree approximation. This is because the dynamics of the

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vacuum value alone forms already a self-contained theory. Then later we wish to add radiation fields to complete the Heisenberg field ψ.

As shown by Lee,[1] the C-number part of ψ in the tree approximation, namely, $\phi(x)$, obeys the classical Euler equation. When $\phi(x)$ is an unstable solution of the Euler equation, the vacuum, in which the extended object is defined, will decay in time. In consequence, ψ will not satisfy the canonical commutation relation. We therefore assume that $\phi(x)$ is at an extremum.

In order for the vacuum to be stable, however, it is not sufficient that $\phi(x)$ is at an extrema. To be stable, the vacuum has to be free from the tunneling process. Thus we assume that $\phi(x)$ has a center and is either symmetrical or antisymmetrical for the total reflection of the space coordinates with respect to this center.

As mentioned previously, the ψ satisfies the conservation relations $[\partial_\mu + iP_\mu, \psi] = 0$ and $[x_\mu\partial_\nu - x_\nu\partial_\mu + im_{\mu\nu}, \psi] = 0$. Correspondingly, we assume that the vacuum value $\langle\psi\rangle$ satisfies the conservation relations $[\partial_\mu + i\langle P_\mu\rangle, \langle\psi\rangle] = 0$ and $[x_\mu\partial_\nu - x_\mu\partial_\nu + i\langle m_{\mu\nu}\rangle, \langle\psi\rangle] = 0$. The validity of such assumptions was verified in case of $1+1$ dimensions by Jacquot,[6] when he investigated the Lorentz transformation property of ψ. The ultimate justification of this assumption will be obtained when we complete the explicit form of the Heisenberg field ψ, instead of just $\langle\psi\rangle$.

The $\partial_\mu\phi(x)$ and $(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi(x)$ do not vanish, since the order parameter $\phi(x)$ is assumed to be local and deformed. Consequently, the infinitesimal operators ∂_μ and $x_\mu\partial_\nu - x_\nu\partial_\mu$ are not conserved by the vacuum value $\langle\psi\rangle$. Then two sets of conservation relations for $\langle\psi\rangle$ mentioned previously indicate that $\langle\psi\rangle$, $\langle P_\mu\rangle$, and $\langle m_{\mu\nu}\rangle$ cannot be C-numbers. The fact that the $\partial_\mu\langle\psi\rangle$ do not vanish but $[\partial_\mu + i\langle P_\mu\rangle, \langle\psi\rangle]$ do vanish implies that the object $\langle\psi\rangle$ fluctuates. This is a well-known quantum-mechanical fluctuation of the position and momenta. (Later we have to distinguish this fluctuation from angular fluctuation. We will call this a linear fluctuation or a translation fluctuation.)

Among the other set of conservation relations, the three relations $[x_\alpha\partial_\beta - x_\beta\partial_\alpha + i\langle m_{\alpha\beta}\rangle, \langle\psi\rangle] = 0$ express the conservation of angular momenta. The $\langle m_{\alpha\beta}\rangle$ are well-known quantum mechanical angular momenta. In conventional quantum mechanics of the point particle, the $\langle m_{\alpha\beta}\rangle$ represent the orbital angular momenta. When one deals with the deformed object, however, the $\langle m_{\alpha\beta}\rangle$ consist of intrinsic and orbital angular momenta, as will be seen in Sec. VI. These intrinsic angular momenta will be denoted as $\hat{S}_\alpha$. In this paper we place the center of the C-number part $\phi(x)$ at the origin. Therefore, when we ignore the linear fluctuation mentioned above, namely, when we replace $\langle P_\alpha\rangle$ by zero, the $\langle m_{\alpha\beta}\rangle$ consist of intrinsic angular momenta only. Then we have a conservation relation $[x_\alpha\partial_\beta - x_\beta\partial_\alpha + i\langle\hat{S}_\gamma\rangle, \langle\psi\rangle] = 0$. They indicate that the deformed object fluctuates angularly around its center. That is to say, it is a superposition of various components, each of which rotates around each different axis with each angular velocity.

The momenta P_μ and angular momenta $m_{\mu\nu}$ are independent of the variables x, y, and z, since they are the values already integrated over these space variables, namely, $\int T_{0\mu}d^3x$ and $\int(x_\mu T_{0\nu} - x_\nu T_{0\mu})d^3x$. Here T is the energy-momentum density. Furthermore these integrals are conserved quantities, and therefore, are independent of time. Then the vacuum values $\langle P_\mu\rangle$ and $\langle m_{\mu\nu}\rangle$ are pure quantum-mechanical matrices. In Sec. VIII, we will construct explicitly the matrix representations of the $\langle P_\mu\rangle$ and $\langle m_{\mu\nu}\rangle$. In quantum mechanics there appear always the canonical conjugates of $\langle P_\alpha\rangle$. In our theory these canonical conjugates, which will be denoted as $\langle\hat{Q}_\alpha\rangle$, are acquired as polynomials of $\langle P_\mu\rangle$ and $\langle m_{\mu\nu}\rangle$ [see (7.3)]. The classical quantum mechanics has always been presented in terms of these $\langle\hat{Q}_\alpha\rangle$ and $\langle P_\mu\rangle$, and is not manifestly relativistic. In this paper we consider $\langle P_\mu\rangle$ and $\langle m_{\mu\nu}\rangle$, or alternatively, $\langle P_\mu\rangle$ and $\langle N_\mu\rangle = \langle P^\nu\rangle\langle m_{\mu\nu}\rangle/\langle P^\sigma\rangle\langle P_\sigma\rangle$ as fundamental quantum-mechanical operators. We regard the quantum mechanics as the polynomial algebra of these fundamental matrices $\langle P_\mu\rangle$ and $\langle N_\mu\rangle$. The quatnum mechanics thus presented is manifestly relativistic.

The quantum matrices discussed so far can be all constructed ultimately by making use of the energy-momentum density T. In our theory, there further appears a set of matrices denoted by R_α, which cannot be constructed in the same manner. They represent the orientation of the deformed object. They are conjugates of $\hat{S}_\alpha$ (but are not canonical conjugates, since R_α and $\hat{S}_\beta$ with $\alpha \neq \beta$ do not commute). The $\langle R_\alpha\rangle$ and $\langle\hat{S}_\beta\rangle$ being quantum matrices, the orientation obeys the uncertainty principle.

We are thus using a number of quantum-mechanical operators in this series of papers. In Sec. VIII, we will examine the way of constructing the matrix representations of these operators. In order to do this we evidently have to introduce some variables in the Hilbert space. As will be shown in Sec. VIII, two sets of variables are needed. One is related principally to $\langle P_\mu\rangle$ and $\langle\hat{Q}_\alpha\rangle$ and is a set of coordinates for the center of $\phi(x)$. The other one is related to $\langle\hat{S}_\alpha\rangle$. The $\langle\hat{S}_\alpha\rangle$ are the matrix representations of the infinitesimal rotations of the deformed object, as one can easily imagine from the conservation relations mentioned previously. In order to describe the rotation of the deformed object, we need to introduce some quantities to indicate its orientation. We will introduce three vectors, which are mutually orthogonal and are fixed on the deformed object. These vectors are the variables which we use in the Hilbert space for $\langle\hat{S}_\alpha\rangle$.

The present work is a continuation of the previous one I. In the next section we will begin by indicating briefly how the zero modes (7.1) of I and the solution (7.8) of I in $2+1$ dimensions have to be modified in case of $3+1$ dimensions. Since we consider only the vacuum value of ψ in this paper, we will denote all the vacuum values $\langle P_\alpha\rangle$, $\langle Q_\alpha\rangle$, $\langle\hat{S}_\sigma\rangle$, $\langle m_{\mu\nu}\rangle$, etc. simply as P_α, Q_α, $\hat{S}_\alpha$, $m_{\mu\nu}$, etc. in the following.

All the notations used in I are carried over into this paper with slight modifications. The ψ^f, ϕ^f, Q, Q^0, $\hat{Q}^0$, $\dot{Q}$, R, S, and $\hat{S}$ are replaced by ψ, ϕ, $-Q$, $-Q^0$, $-\hat{Q}^0$, $-\dot{Q}$, $-R$, $-S$, and $-\hat{S}$ in this paper. Also the opposite sign is adopted for the Lagrangian.

II. ZERO MODE

The content of this section is an extension of the result

given in Sec. VII of I, and the detail of the argument can be found there.

In the case of 2 + 1 dimensions, the zero energy modes are given in (7.1) of I. Exactly in the same way, the zero energy modes in 3 + 1 dimensions are given as

$$\partial_\mu \phi(x) \quad \text{and} \quad (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x), \tag{2.1}$$

where μ and ν run over x, y, z, and t [in (62) of I, they cover x, y, and t].

Then the conserved angular momenta are

$$(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + i m_{\alpha\beta}, \tag{2.2}$$

where

$$m_{\alpha\beta} = \int (x_\alpha T_{0\beta} - x_\beta T_{0\alpha})\, dx\, dy\, dz \tag{2.3}$$

and $T_{\mu\nu}$ are components of the energy momentum density. The α and β run over x, y, and z. Throughout this paper we use μ, ν, σ, and ρ for x, y, z, and t and α, β, and γ for x, y, and z only. The fact that the conserved angular momenta are not just $(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$, but include $i m_{\alpha\beta}$, indicated that the nonspherical object fluctuates angularly around the axis parallel to the vector $m_{\alpha\beta}$ as was mentioned in the previous section. Then the function $\phi(x, y, t, R^0, S)$, given in (7.5) of I [this function is written as $\phi(x, y, t, S)$ in I], has to be extended into the following form in the case of 3 + 1 dimensions:

$$\phi(x,y,z,t,R^0,S) = \exp(-(R^0_\alpha + S_\alpha t)(x_\beta \partial_\gamma - x_\gamma \partial_\beta))\rho(x,y,z,s), \tag{2.4}$$

where $(R^0 + S_\alpha t)(x_\beta \partial_\gamma - x_\gamma \partial_\beta)$ means

$$(R^0_x + S_x t)(y\partial_z - z\partial_y) + \text{cyclic permutations}. \tag{2.5}$$

In order that the $\phi(x, y, z, t, R^0, S)$ satisfies the Euler equation (2.2) of I, the ρ has to satisfy the equation

$$-\sum_\alpha \partial_\alpha^2 \rho + m^2 \rho + \{S_\alpha (x_\beta \partial_\gamma - x_\gamma \partial_\beta)\}^2 \rho = F(\rho). \tag{2.6}$$

The S_x, S_y, and S_z do not transform like three components of a vector as will be seen later. The ρ depends on S_α. In fact, however, the dependency of ρ on S_α can be represented simply by the length of S_α, denoted by s. (In order to make this point clear, we have to wait till near the end of Sec. VI. In particular see Ref. 19.) Therefore we have expressed ρ as $\rho(x, y, z, s)$ in (2.4).

The functions $\phi(x, y, z, t, R^0, S)$ and $\rho(x, y, z, s)$ are equal to the order parameter $\phi(x)$ when $R^0_\alpha = S_\alpha = 0$. Therefore $\phi(x, y, z, t, R^0, S)$ can be expanded in the following way:

$$\phi(x,y,z,t,R^0,S) = \phi - (R^0_\alpha + S_\alpha t)(x_\beta \partial_\gamma - x_\gamma \partial_\beta)\phi + S_\alpha \left(\frac{\partial}{\partial S_\alpha}\rho\right)_{S=0} + \text{higher order terms}. \tag{2.7}$$

Here ϕ means the order parameter $\phi(x)$.

Then, when the Heisenberg field ψ is expanded in terms of the quantum operators and the creation and annihilation operators, its linear term $\psi^{(1)}$ becomes

$$\psi^{(1)} = -Q^0 \partial\phi - t\dot{Q}^\alpha (x_\alpha \partial_t + t\partial_\alpha)\phi - (R^0_\alpha + S_\alpha t)(x_\beta \partial_\gamma - x_\gamma \partial_\alpha)\phi + S_\alpha \left(\frac{\partial}{\partial S_\alpha}\rho\right)_{S=0} + \sum \chi_i + \int \chi_k. \tag{2.8}$$

III. THE SOLUTION

There are infinitely many solutions of the Heisenberg equation [(3) of I], having $\psi^{(1)}$ [(2.8)] as its linear term, and yet the values of their energies are not all identical and consequently these solutions are not always related to each other by the time-independent canonical transformation. The situation is exactly the same in the case of 2 + 1 dimensions and was explained in detail in I. The purpose of this paper is to find among these solutions a solution which satisfies the canonical commutation relation and therefore is Lorentz covariant.

We start from the following solution,

$$\exp(-N^\mu \partial_\mu)\exp(-A^\alpha (t\partial_\alpha + x_\alpha \partial_t)) \times \exp(-(R^0_\alpha + S_\alpha t)(x_\beta \partial_\gamma - x_\gamma \partial_\beta))\rho(x,y,z,s). \tag{3.1}$$

The order of the operators N, A, R^0, and S are defined as follows. Expand the solutions (8.3) in a power series of N, A, R^0, and S. Then symmetrize the order of these operators at each term in the series. Since ∂_μ and $x_\mu \partial_\nu - x_\nu \partial_\mu$ are infinitesimal operators of the inhomogeneous Lorentz group, this function satisfies the Heisenberg equation. Further this has $\psi^{(1)}$ [(2.8)] as its linear term.

In terms of the quantities X and T introduced in the previous paper [(8.5) and (8.6) of I], the solution (3.1) can be expressed as

$$\exp(-(R^0_\alpha + S_\alpha T)(X_\beta \partial_\gamma - X_\gamma \partial_\beta))\rho(X,Y,Z,s), \tag{3.2}$$

where $\partial_\alpha = \partial/\partial X^\alpha$. As will be shown later in Sec. VI, the N_μ satisfy the relation $P^\mu N_\mu = 0$ [see (6.8)]. As a result, T becomes independent of N_μ and the expression (13.18) of I is still valid for T in this paper.

If we introduce the coordinates X' and T',

$$X' = \exp(-N^\mu \partial_\mu)\exp(-A^\alpha (t\partial_\alpha + x_\alpha \partial_t)) \times \exp(-(R^0_\alpha + S_\alpha t)(x_\beta \partial_\gamma - x_\gamma \partial_\beta))x, \qquad T' = T, \tag{3.3}$$

the ϕ becomes simply ρ.

$$\phi = \rho(X', Y', Z', s). \tag{3.4}$$

In the following we occasionally denote the solution (3.1) [and therefore (3.2) and (3.4)] as

$$\phi(X,Y,Z,T,R^0,S). \tag{3.5}$$

IV. ENERGY AND MOMENTUM

The energy of the extended object ϕ can be calculated in the same way as in the case of 2 + 1 dimensions (see Sec. X of I). After some cumbersome calculation we obtain the result

$$H = -P_0 = -\int T_{00}\, d^3x = \frac{s^2 I + M(s)}{(1-\dot{Q}^2)^{1/2}} = \frac{\hat{M}}{(1-\dot{Q}^2)^{1/2}}, \tag{4.1}$$

where

$$\dot{Q}_\alpha = \tanh A_\alpha, \tag{4.2}$$

$$I = \frac{1}{2}\int ((x\times\partial)_S\phi)^2\, d^3x = \frac{1}{2}\int ((x\times\partial)_S\rho)^2\, d^3x, \tag{4.3}$$

$$M(s) = \frac{1}{3}\int \sum(\partial_x\phi)^2\, d^3x = \frac{1}{3}\int \sum(\partial_x\rho)^2\, d^3x, \tag{4.4}$$

$$\hat{M} = s^2 I + M(s). \tag{4.5}$$

In (4.3), the $(x\times\partial)_S$ indicates the infinitesimal rotation around the direction of S, namely,

$$(x\times\partial)_S = (S_\alpha/|S|)(x_\beta\partial_\gamma - x_\gamma\partial_\beta). \tag{4.6}$$

The ϕ appearing in (4.3) and (4.4) is $\phi(x,y,z,t,R^0,S)$, namely (2.4), or equivalently (2.7). It depends on t, but $M(s)$ becomes independent of t after integration.

As it will be shown later in this section, the energy of the extended object (4.1) can be rewritten as

$$H = (\tfrac{1}{2}S^2 I + M'(s))/(1-\dot{Q}^2)^{1/2}, \tag{4.7}$$

where

$$M'(s) = \int \frac{1}{2}\sum\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}m^2\phi^2 + V(\phi)\, d^3x$$
$$= \int \frac{1}{2}\sum\left(\frac{d\rho}{dx}\right)^2 + \frac{1}{2}m^2\rho^2 + V(\rho)\, d^3x.$$

Then the mass (4.5) can be expressed as

$$\hat{M} = \tfrac{1}{2}S^2 I + M'(s). \tag{4.8}$$

In deriving expression (4.1), we have used the following relation for $\phi(x,y,z,t,R^0,S)$:

$$\int\left(\frac{d\phi}{dx}\right)^2 d^3x = \int\left(\frac{d\phi}{dy}\right)^2 d^3x = \int\left(\frac{d\phi}{dz}\right)^2 d^3x. \tag{4.9}$$

We can derive these relations from the variational principle

$$\delta\int\left\{\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{1}{2}\sum\left(\frac{d\phi}{dx}\right)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right\}d^3x = 0. \tag{4.10}$$

Choosing $\delta\phi$ to be

$$\delta\phi = \phi(x+\epsilon x,y,z,t,R^0,S) - \phi(x,y,z,t,R^0,S), \tag{4.11}$$

we obtain

$$\int\left\{\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - \frac{1}{2}\left(\frac{d\phi}{dy}\right)^2 - \frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right\}d^3x = 0. \tag{4.12}$$

From the further two $\delta\phi$, namely,

$$\delta\phi = \phi(x,y+\epsilon y,z,t,R^0,S) - \phi(x,y,z,t,R^0,S)$$

and

$$\delta\phi = \phi(x,y,z+\epsilon z,t,R^0,S) - \phi(x,y,z,t,R^0,S),$$

we obtain two other equations similar to (4.12). From these three equations, we can easily obtain the relation (4.9).

The momenta of the solution (3.1) can be calculated in the same manner:

$$P_x = \int T_{0x}\, d^3x = \frac{\dot{Q}_x\hat{M}}{(1-\dot{Q}^2)^{1/2}}. \tag{4.13}$$

Therefore the momenta P are related to the energy H by the relations

$$P_\alpha = \dot{Q}_\alpha H = -\dot{Q}_\alpha P_0. \tag{4.14}$$

Later in Sec. VIII we need to know the values dH/ds and dP/ds. In order to find these values, we first construct the Lagrangian from which we can deduce Eq. (2.6) for ρ. Then the function ρ satisfies the variational relation

$$\delta\int\left\{\frac{1}{2}S_\alpha^2((x_\beta\partial_\gamma - x_\gamma\partial_\beta)\rho)^2 - \frac{1}{2}\sum(\partial_x\rho)^2 - \frac{1}{2}m^2\rho^2 - V(\rho)\right\}d^3x = 0. \tag{4.15}$$

Choosing $\delta\rho$ to be

$$\delta\rho = \rho(x,y,z,s+\delta) - \rho(x,y,z,s), \tag{4.16}$$

we obtain the relation

$$\frac{1}{2}s^2\frac{d}{ds}\int((x\times\partial)_S\rho)^2 d^3x - \frac{d}{ds}\left\{\frac{1}{2}\sum(\partial_x\rho)^2 + \frac{1}{2}m^2\rho^2 + V(\rho)\right\}d^3x = 0. \tag{4.17}$$

Choosing $\delta\rho$ to be

$$\delta\rho = \rho(x+\epsilon x,y+\epsilon y,z+\epsilon z,s) - \rho(x,y,z,s), \tag{4.18}$$

we obtain

$$\int\left\{\frac{3}{2}((x\times\partial)_S\rho)^2 - \frac{1}{2}\sum(\partial_x\rho)^2 - \frac{3}{2}m^2\rho^2 - 3V(\rho)\right\}d^3x = 0. \tag{4.19}$$

Expression (4.7) for the energy can be derived from the original one, (4.1), by using relation (4.19). Then the $d\hat{M}/ds$ can be obtained by using the relation (4.17) as

$$\frac{d}{ds}\hat{M} = \frac{d}{ds}\left(\frac{1}{2}s^2 I + M(s)\right) = \frac{d}{ds}\left(s^2 I - \frac{1}{2}s^2 I + M(s)\right)$$
$$= \frac{d}{ds}(s^2 I) - sI = s\frac{d(sI)}{ds}. \tag{4.20}$$

Finally, we obtain the following values for dH/ds and dP/ds:

$$\frac{d}{ds}H = \frac{s}{(1-\dot{Q}^2)^{1/2}}\frac{d\hat{s}}{ds}, \tag{4.21}$$

$$\frac{d}{ds}P_\alpha = \frac{\dot{Q}_\alpha s}{(1-\dot{Q}^2)^{1/2}}\frac{d\hat{s}}{ds}, \tag{4.22}$$

where $\hat{s} = sI = (\Sigma_\alpha S_\alpha^2)^{1/2} I$.

V. ANGULAR MOMENTUM TENSOR $m_{\mu\nu}$

Lengthy and complicated calculations are needed to obtain $m_{\mu\nu}$. However, the result is rather simple,

$$m_{0x} = \int (tT_{0x} - xT_{00}) d^3x$$
$$= -\int \{x - N_x - \dot{Q}_x(t - N_0)\} T_{00} d^3x$$
$$- (N_x - \dot{Q}_x N_0) \int T_{00}$$
$$= -(N_x - \dot{Q}_x N_0) P_0 + \frac{\dot{Q}_y \hat{S}_z - \dot{Q}_z \hat{S}_y}{(1 - \dot{Q}^2)^{1/2}}$$
$$= [N \times P]_{0x} + \frac{[\dot{Q} \times \hat{S}]_{yz}}{(1 - \dot{Q}^2)^{1/2}}, \tag{5.1}$$

$$m_{xy} = \int (xT_{0y} - yT_{0x}) d^3x$$
$$= \hat{S}_z + \frac{1}{\dot{Q}^2}\left(-1 + \frac{1}{(1 - \dot{Q}^2)^{1/2}}\right)\left(\sum_i \dot{Q}_y \dot{Q}_i \hat{S}_{xi}\right.$$
$$\left. - \sum_i \dot{Q}_x \dot{Q}_i \hat{S}_{yi}\right) + (Q_x P_y - Q_y P_x)$$
$$= \frac{\hat{S}_z}{(1 - \dot{Q}^2)^{1/2}} - \left(-1 + \frac{1}{(1 - \dot{Q}^2)^{1/2}}\right)$$
$$\times \frac{\dot{Q}_z(\dot{Q} \cdot \hat{S})}{\dot{Q}^2} + [N \times P]_{xy}. \tag{5.2}$$

In the above $\hat{S}_\alpha = S_\alpha I_\alpha$ with $I_\alpha = \int (x_\beta \partial_\gamma - x_\gamma \partial_\beta)\phi)^2 d^3x$.[7] The $(\dot{Q} \cdot \hat{S})$ means $\dot{Q}_x \hat{S}_x + \dot{Q}_y \hat{S}_y + \dot{Q}_z \hat{S}_z$.

In the course of the calculation above, the following three relations for $\phi(x, y, z, t, R^0, S)$ are used:

$$\int (x_\alpha \partial_\beta + x_\beta \partial_\alpha)\phi \, (x \times \partial)_S \phi \, d^3x = 0, \tag{5.3}$$

where $(x \times \partial)_S$ is defined in (4.6). These three relations can be derived as follows.

We first rotate the coordinate system x, y, and z into a new coordinate system x', y', and z' so that the new third axis z' coincides with the direction of the angular velocity S. Then the $(x \times \partial)_S$ is expressed in the new coordinate system simply as $(x \times \partial)_S = (x' \partial_{y'} - y' \partial_{x'})$. Consequently, required relations (5.3) are written as

$$\int (x'_\alpha \partial'_\beta + x'_\beta \partial'_\alpha)\phi \, (x' \partial_{y'} - y' \partial_{x'})\phi \, d^3x' = 0. \tag{5.4}$$

The variational principle $\delta \int \mathscr{L} \, d^3x = 0$ has to be satisfied for any infinitesimal variation $\delta\phi$ vanishing at infinity. Choosing the following $\delta\phi$,

$$\delta\phi = \phi(x' + \epsilon x', y' - \epsilon y', z', t, S)$$
$$- \phi(x', y', z', t, S),$$

and taking into account the relations (4.9), we can derive from $\delta \int d^3x = 0$ one of three relations (5.4), namely,

$$\int (x' \partial_{y'} + y' \partial_{x'})\phi \, (x' \partial_{y'} - y' \partial_{x'})\phi \, d^3x = 0. \tag{5.5}$$

Let us choose the following $\delta\phi$:

$$\delta\phi = \phi(x', y' + \epsilon z', z', t, R^0, S) - \phi(x', y', z', t, R^0, S).$$

Then the $\delta \int \mathscr{L} \, d^3x$ can be calculated easily by changing the variables from x', y', and z' into $x'' = x'$, $y'' = y' + \epsilon z'$, and $z'' = z'$. This is because the measure of integration is not altered, namely, $dy'' \, dz'' = dy' \, dz'$. Also, $\int (\partial_{y'} \phi)^2 d^3x$ and $\int (\partial_{z'} \phi)^2 d^3x$ are not altered by this change of variables since $\int \partial_{y'} \phi \partial_{z'} \phi \, d^3z = 0$. Then, from $\delta \int \mathscr{L} \, d^3x = 0$, we obtain the relation

$$\int z' \partial_{x'} \phi \, (x' \partial_{y'} - y' \partial_{x'})\phi \, d^3x = 0. \tag{5.6}$$

In the same way, from $\delta\phi = \phi(x', y', z' + \epsilon y', t, R^0, S) - \phi(x', y', z', t, R^0, S)$, we obtain

$$\int x' \partial_{z'} \phi \, (x' \partial_{y'} - y' \partial_{x'})\phi \, d^3x = 0. \tag{5.7}$$

From the two relations (5.6) and (5.7), we obtain another relation among the relations (5.4),

$$\int (z' \partial_{x'} + x' \partial_{z'})\phi \, (x' \partial_{y'} - y' \partial_{x'})\phi \, d^3x = 0. \tag{5.8}$$

In the same way we can derive the remaining one, namely, (5.4) with $\alpha = y$ and $\beta = z$, if one adopts the following $\delta\phi$:

$$\delta\phi = \phi(x' + \epsilon z, y', z' + \epsilon x', t, R^0, S)$$
$$- \phi(x', y', z', t, R^0, S). \tag{5.9}$$

In (5.1) and (5.2), the angular momenta $m_{\mu\nu}$ are obtained in terms of N, $\dot{Q}$, and $\hat{S}$. We can express $m_{\mu\nu}$ in another form by introducing the following operator B,

$$B_\alpha = N_\alpha + \dot{Q}_\alpha\left(-1 + \frac{1}{(1 - \dot{Q}^2)^{1/2}}\right)$$
$$\times \frac{(\dot{Q} \cdot N)}{\dot{Q}^2} - \frac{\dot{Q}_\alpha N_0}{(1 - \dot{Q}^2)^{1/2}}. \tag{5.10}$$

The nature of this operator will become clear later. Then in terms of operators B, $\dot{Q}$, and $\hat{S}$, the angular momenta $m_{\mu\nu}$ are expressed as

$$m_{0\alpha} = \frac{B_\alpha \hat{M}}{(1 - \dot{Q}^2)^{1/2}} - \dot{Q}_\alpha\left(-1 + \frac{1}{(1 - \dot{Q}^2)^{1/2}}\right)$$
$$\times \frac{(\dot{Q} \cdot B)\hat{M}}{\dot{Q}^2} - \frac{[\dot{Q} \times \hat{S}]_{\beta\gamma}}{(1 - \dot{Q}^2)^{1/2}}, \tag{5.11}$$

$$m_{\alpha\beta} = \frac{\hat{S}_\gamma}{(1 - \dot{Q}^2)^{1/2}} - \left(-1 + \frac{1}{(1 - \dot{Q}^2)^{1/2}}\right)$$
$$= \quad \times \frac{\dot{Q}_\gamma(\dot{Q} \cdot \hat{S})}{\dot{Q}^2} + [B \times P]_{\alpha\beta}$$
$$= \frac{\hat{S}_\gamma}{(1 - \dot{Q}^2)^{1/2}} - \left(-1 + \frac{1}{(1 - \dot{Q}^2)^{1/2}}\right)$$
$$\times \frac{\dot{Q}_\gamma(\dot{Q} \cdot S)}{\dot{Q}^2} + \frac{[B \times \dot{Q}]_{\alpha\beta}\hat{M}}{(1 - \dot{Q}^2)^{1/2}}. \tag{5.12}$$

We now compare these two expressions with the formula of the Lorentz transformation for the antisymmetrical tensors $\mathscr{T}_{\mu\nu}$. Let us assume that the coordinate system A' moves with a velocity v relative to the rest system A. Let us denote these antisymmetrical tensors observed in the system A' by $\mathscr{T}'_{\mu\nu}$. We denote $\mathscr{T}_{0\alpha}$ and $\mathscr{T}'_{0\alpha}$ by $\mathscr{C}_\alpha$ and $\mathscr{C}'_\alpha$, and also $\mathscr{T}_{\alpha\beta}$ and $\mathscr{T}'_{\alpha\beta}$ by $\mathscr{H}_\alpha$ and $\mathscr{H}'_\alpha$. Then the antisymmetrical tensors $\mathscr{T}_{\mu\nu}$ are transformed as follows:

$$\mathscr{E}_\alpha = \frac{\mathscr{E}'_\alpha}{(1-v^2)^{1/2}} + \frac{v_\alpha}{v^2}(v\cdot\mathscr{E}') \times\left(1-\frac{1}{(1-v^2)^{1/2}}\right) - \frac{[v\times\mathscr{H}']_{\beta\gamma}}{(1-v^2)^{1/2}}, \tag{5.13}$$

$$\mathscr{H}_\alpha = \frac{\mathscr{H}'_\alpha}{(1-v^2)^{1/2}} + \frac{v_\alpha}{v^2}(v\cdot\mathscr{H}')\left(1-\frac{1}{(1-v^2)^{1/2}}\right) - \frac{[v\times\mathscr{E}']_{\beta\gamma}}{(1-v^2)^{1/2}}. \tag{5.14}$$

These two relations (5.13) and (5.14) are identical with the two expressions (5.11) and (5.12) if we replace $\mathscr{E}$, $\mathscr{H}$, and v by $B\hat{M}$, $\hat{S}$, and $-\dot{Q}$. Therefore, if we observe angular momentum tensors in the coordinate system moving with velocity $\dot{Q}$, we will find the values $B_\alpha\hat{M}$ and $\hat{S}_\alpha$. This fact can be explained also in another way.

One may formulate the theory of the extended object in the fluctuating coordinate system. If we describe the extended object in the coordinate system which is moving with matrix velocity $\dot{Q}$, the Heisenberg field becomes independent of $\dot{Q}$. The position operators still remain there, but are no longer Q^0. The static extended object observed in the fluctuating coordinate system becomes a function of $x-B$ instead of X, as will be explained below.

The coordinates X introduced in (6.3) and (8.5) of I can be written in terms of B as

$$X = \hat{X} - B, \tag{5.15}$$

where $\hat{X}$ are the coordinates given in (6.10) of I and are functions of x, y, z, t and $\dot{Q}$. In the fluctuating coordinate system $\hat{X}$ becomes simply x. Consequently the static extended object which is originally a function of X, becomes a function of $x-B$. The deformed object depends further on T. However, T becomes t in the fluctuating coordinate system. Therefore the deformed object becomes simply a function of $x-B$ and t.

Now the values of the angular momenta $m_{0\alpha}$ in the fluctuating coordinate system can be calculated as follows

$$\int (tT'_{0x} - xT'_{00})d^3x = -\int (x-B)T'_{00}\,d^3x + t\int T'_{0x}\,d^3x - B\int T'_{00}\,d^3x. \tag{5.16}$$

The tensor T' is the energy-momentum density of the object observed in the fluctuating coordinate system, and is a function of $x-B$ (in the case of the deformed object, further of t, R^0, and S). The first term is a constant and we ignore it.[8] The second term vanishes since the momentum of the object at rest is zero. Finally we obtain the value $B\hat{M}$.

The other components of the angular momentum, namely $m_{\alpha\beta}$ can be calculated in the same manner. One can easily confirm that they are $\hat{S}_\gamma$.

VI. LORENTZ COVARIANCE (1)

The purpose of this paper is to find, among infinitely many solutions of the Euler equation, a solution which is Lorentz covariant. In this section we will prove that the relativistic angular momenta $m_{\mu\nu}$ [(5.1), (5.2)] form an infinitesimal Lorentz group. More precisely, if we choose suitable commutation relations for the quantum mechanical operators N_μ, A_α, (or $\dot{Q}_\alpha$), and $\hat{S}_\alpha$, the $m_{\mu\nu}$ satisfy the commutation relations

$$i[m_{\mu\nu}, m_{\sigma\rho}] = -g_{\mu\sigma}m_{\nu\rho} + g_{\mu\rho}m_{\gamma\sigma} + g_{\nu\sigma}m_{\mu\rho} - g_{\nu\rho}m_{\mu\gamma}. \tag{6.1}$$

In fact, however, such required commutation relations among operators N_μ, A_α (or $\dot{Q}_\alpha$), and $\hat{S}_\alpha$ are not simple, and it is not worth presenting them. Instead there is a simple and clear way of determining these operators themselves directly in a self-consistent manner (the meaning of words self-consistent manner will become clear soon). We will do this in the following.

Ten operators P_μ and $m_{\mu\nu}$, given in formulas (4.1), (4.13), (5.1), and (5.2), are written in terms of operators N_μ, P_μ, and $\hat{S}_\alpha$. Conversely, we can express N_μ and $\hat{S}_\alpha$ in terms of P_μ and $m_{\mu\nu}$. This is because only ten operators among N_μ, P_μ, and $\hat{S}_\alpha$ are independent, because of the relation $P^\mu N_\mu = 0$, which will be proven later. We first express N_μ in terms of P_μ and $m_{\mu\nu}$. From two expressions (5.1) and (5.2), we obtain the relations

$$P^\nu m_{\nu\mu} = \hat{M}^2 N_\mu - P_\mu(P^\nu N_\nu). \tag{6.2}$$

The solutions N_μ of this equation are not unique. The general solutions are

$$N_\mu = P^\nu m_{\nu\mu}/\hat{M}^2 + aP_\mu, \tag{6.3}$$

where a is an arbitrary constant. Now we deal with $\hat{S}$. From (5.2) we obtain the relation

$$\sum_\alpha \dot{Q}^\alpha m_\alpha = \sum_\alpha \dot{Q}^\alpha \hat{S}_\alpha, \tag{6.4}$$

where $m_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}m^{\beta\gamma}$. By inserting the expressions (6.4) and (6.3) into the relations (5.2), we obtain $\hat{S}_\alpha$ as follows

$$\hat{S}_\alpha = 1/\hat{M}\left\{W_\alpha - \dot{Q}_\alpha\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right) \times \frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}(\dot{Q}\cdot W)\right\}, \tag{6.5}$$

where

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}P^\nu m^{\sigma\rho}. \tag{6.6}$$

Conversely W can be expressed in terms of $\hat{S}$ as[9]

$$\frac{W_\alpha}{M} = \hat{S}_\alpha + \dot{Q}_\alpha\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{(\dot{Q}\cdot\hat{S})}{\dot{Q}^2}. \tag{6.7}$$

The N_μ have not been determined uniquely yet, since a in (6.3) is arbitrary. We fix the value of a by requiring that the solution (3.1) with N_μ given by (6.3), reduces to $\phi(x,y,z,t,R^0, S)$ [(2.4)], when $\dot{Q}_\alpha$ vanish.[10] Then we obtain the following expressions for N_μ.

$$N_\mu = P^\nu m_{\nu\mu}/\hat{M}^2. \tag{6.8}$$

The N_μ [(6.8)] satisfy the relation $P^\mu N_\mu = 0$. The set $N = (N_0, N_x, N_y, N_z)$ is evidently a relativistic vector. It expresses how far the trajectory of the center of mass deviates from the origin, as will be explained in Sec. XI. Therefore we will call it the relativistic displacement operator.

Now we explain the situation where we are. Let us consider a quantum mechanical particle with mass $\hat{M}$, which is

moving freely in the space-time of the coordinates t, ξ_x, ξ_y, and ξ_z (see Sec. XI). We then construct the matrix representations of the infinitesimal operators of the inhomogeneous Lorentz group, namely P_μ and $m_{\mu\nu}$ (see Secs. VIII and XI). Out of these operators, we construct A_α, N_μ, and $\hat{S}_\alpha$ by the relations (4.2), (4.14), (6.5), and (6.8). Thus the solution (3.1) is now determined. (The operators R are not determined yet. These are irrelevant to the matter in the discussion in this section). Evidently the energy-momentum vector and angular momentum tensor of the solution thus fixed, are again P_μ and $m_{\mu\nu}$ themselves, namely, the matrix representations of the infinitesimal inhomogeneous Lorentz group. Now everything is in order.

In the previous paper I we have used three quantum-mechanical operators denoted as Q_α. In terms of P_μ and $m_{\mu\nu}$ these operators can be written as

$$Q_\alpha = Q^0{}_\alpha + \dot{Q}_\alpha t = N_\alpha - \dot{Q}_\alpha N_0 + \dot{Q}_\alpha t = \frac{P^\nu m_{\mu\nu}}{P^\sigma P_\sigma} + \frac{P_\alpha}{P^0}\left(-\frac{P^\nu m_{0\nu}}{P^\sigma P_\sigma} + t\right). \tag{6.9}$$

The $Q^0{}_\alpha = N_\alpha - \dot{Q}_\alpha N_0$ are not relativistic quantities. Nevertheless, as will be shown in Sec. XI, Q_α are relativistic and represent the center of mass (more precisely, center of energy) of the extended object. The idea of constructing the center of mass, either classical or quantum mechanical, out of the energy-momentum vectors and the angular momentum tensors, is not new. In the case of classical relativity theory, it was tried many years ago already by Fokker.[11] As to the quantum-mechanical center of mass, we may refer to Pryce's paper which is one of the earliest and yet is already complete.[12] The definition of the relativistic center of mass, however, has not been unique (for example, see Sec. I of Ref. 12). There have been several proposals. Our definition agrees with the one given by Pryce [see (II.5)]. It is convenient to introduce the following notation V_μ:

$$V_\mu = P^\nu m_{\mu\nu} = -\hat{M}^2 N_\mu. \tag{6.10}$$

Then two Lorentz vectors W_μ and V_μ satisfy commutation relations.

$$\begin{aligned} &[W_\mu, W_\nu] = \tfrac{1}{2} i\epsilon_{\mu\nu\sigma\rho}(P^\sigma W^\rho - P^\rho W^\sigma), \\ &[V_\mu, V_\nu] = -i\hat{M}^2 m_{\mu\nu}, \\ &[V_\mu, W_\nu] = -iP_\nu W_\mu, \\ &[W_\mu, P_\nu] = 0. \end{aligned} \tag{6.11}$$

We can then prove the following commutation relations from the relations (6.11),

$$[\hat{S}_\alpha, \hat{S}_\beta] = -\epsilon_{\alpha\beta\gamma}\hat{S}_\gamma. \tag{6.12}$$

Thus the quantum-mechanical operators $\hat{S}_\alpha$ are the representations of the rotation group.[13] They are not irreducible however. They are the sum of irreducible representations, each of which is characterized by an integer l, as will be shown in Sec. VIIIB. Consequently, the real extended object is a particle with mixed intrinsic angular momenta.

VII. CANONICAL CONJUGATE OF P

The operators $N_\alpha - \dot{Q}_\alpha N_0$, which appear in m_{0x} [(5.1)] and are denoted as Q^0 in I, satisfy the commutation relations

$$[N_\alpha - \dot{Q}_\alpha N_0, P_\beta] = i\delta_{\alpha\beta}. \tag{7.1}$$

In case of a spherical extended object, these operators commute among themselves and therefore are the canonical conjugates of P_α [see (9.2) of I]. However, these operators $N_\alpha - \dot{Q}_\alpha N_0$ do not commute among themselves when the extended object is not spherical. Instead they satisfy the commutation relations

$$[N_\alpha - \dot{Q}_\alpha N_0, N_\beta - \dot{Q}_\beta N_0] = -\epsilon_{\alpha\beta\gamma} W_\gamma / H\hat{M}. \tag{7.2}$$

Therefore $N_\alpha - \dot{Q}_\alpha N_0$ are not the canonical conjugates of P.

We denote by $\hat{Q}^0$ the three operators which commute among themselves and reduce to $N_\alpha - \dot{Q}_\alpha N_0$ when W_α vanish.

They are

$$\hat{Q}^0_\alpha = N_\alpha - \dot{Q}_\alpha N_0 + \left(-1 + \frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{[P\times\hat{S}]}{P^2}. \tag{7.3}$$

As will be seen later in Sec. VIII, $\hat{S}$ commutes with $\dot{Q}$. Then we can replace $N_\alpha - \dot{Q}_\alpha N_0$ by $\hat{Q}^0{}_\alpha$ in the commutation relation (7.1). Therefore $\hat{Q}^0{}_\alpha$ are the canonical conjugates of P_α and satisfy the commutation relations

$$[\hat{Q}^0{}_\alpha, \hat{Q}^0{}_\beta] = 0, \quad [\hat{Q}^0{}_\alpha, P_\beta] = i\delta_{\alpha\beta}. \tag{7.4}$$

We will see later in Sec. VIII that the operators $\hat{Q}_\alpha = \hat{Q}^0{}_\alpha + \dot{Q}_\alpha t$ are the Heisenberg representations of the center of the object (not the center of mass).

In terms of the operators $\hat{Q}^0$ the angular momentum $m_{\mu\nu}$ can be expressed as

$$m_{0\alpha} = H\hat{Q}^0{}_\alpha + \left(-1 + \frac{1}{(1-\dot{Q}^2)^{1/2}}\right) \times \frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}[\dot{Q}\times\hat{S}]_{\beta\gamma}, \tag{7.5}$$

$$m_{\beta\gamma} = \hat{S}_\alpha + [\hat{Q}^0\times P]_{\beta\gamma}. \tag{7.6}$$

These two expressions are more convenient than the expressions (5.1) and (5.2) when we prove the Lorentz covariance of ϕ in Sec. IX.

Let us write $H\dot{Q}_\alpha$ in (7.5) $[\hat{Q}^0\times P]_{\alpha\beta}$ in (7.6) as $\hat{m}_{0\alpha}$ and $\hat{m}_{\alpha\beta}$, namely,

$$\hat{m}_{0\alpha} = H\hat{Q}^0{}_\alpha, \tag{7.7}$$

$$\hat{m}_{\alpha\beta} = [\hat{Q}^0\times P]_{\alpha\beta}. \tag{7.8}$$

We can easily confirm that these $\hat{m}_{\mu\nu}$ satisfy the commutation relations (6.1). As will be explained in Secs. VIII and XI, $\hat{Q}^0{}_\alpha$ are matrix representations of the coordinates ξ_α, which represent the center of the extended object. Consequently, $\hat{m}_{\mu\nu}$ are six components of the relativistic orbital angular momentum tensor.

We further introduce operators $\hat{N}_\mu$,

$$\hat{N}_\mu = P^\nu \hat{m}_{\nu\mu}. \tag{7.9}$$

Then $\hat{Q}^0{}_\alpha$ and $\hat{m}_{\mu\nu}$ can be expressed as

$$\hat{Q}^0{}_\alpha = \hat{N}_\alpha - \dot{Q}_\alpha \hat{N}_0, \tag{7.10}$$

$$\hat{m}_{\mu\nu} = [\hat{N}_\mu P_\nu - \hat{N}_\nu P_\mu]. \tag{7.11}$$

These two expressions (7.10) and (7.11) are identical to (9.2) and (9.9) of I for the spherical object.

The commutation relations between $\hat{S}$ and $\hat{Q}^0$ can be

deduced by making use of the relation (6.11). We find that $\hat{S}$ and $\hat{Q}^0$ commute,

$$[\hat{S}_\alpha, \hat{Q}^0{}_\beta] = 0. \tag{7.12}$$

Also, from (6.11), we can easily prove that $\hat{S}$ commute with $\dot{Q}$, and therefore with $\hat{Q}$,

$$[\hat{S}_\alpha, \hat{Q}_\beta] = 0. \tag{7.13}$$

VIII. RELATIVISTIC QUANTUM MECHANICS

The quantum mechanics created in the 1920's was based upon the Hamiltonian formalism. The quantization condition was introduced for the canonical set of variables, namely, the momentum and the position. Neither the Hamiltonian formalism nor such quantization condition is not manifestly relativistic. The theory of quantum mechanics has suffered from its nonrelativistic appearance since then.

In this paper we construct the quantum mechanics in terms of the momenta P_μ and angular momenta $m_{\mu\nu}$. The position operators Q_α appearing in the conventional quantum mechanics can be expressed as polynomials of P_μ, $m_{\mu\nu}$, and $1/P_0$ [(9.2) of I are valid also in this paper]. The quantum mechanics thus presented is manifestly relativistic.

In Subsec. A we treat the quantum mechanics representing the dynamics of the center of mass of the spherical extended object. In this case P_μ and $m_{\mu\nu}$ are the representations of the inhomogeneous Lorentz group acting upon the center (in the case of the spherical object, the center of the object coincides with the center of mass). The $m_{\mu\nu}$ appearing here are restricted to the ones satisfying $W_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}P^\nu m^{\sigma\rho} = 0$. The conventional quantum mechanics of $Q^0{}_\alpha$ and P_μ can be looked at as the polynomial algebra of P_μ and $m_{\mu\nu}$ (or equivalently P_μ and N_μ).

In Subsec. B we consider the case of the nonspherical object. There the quantum mechanics is not simply of the point. It represents the dynamics (rotational) of the orientation of the object as well as the dynamics of the center of mass. The P_μ and $m_{\mu\nu}$ still form the inhomogeneous Lorentz group, but they represent more involved infinitesimal operations. They include spin operators (integers) and therefore W_μ do not vanish. In the wave function (3.1), there appear further operators $R^0{}_\alpha$. The algebra of P_μ, $m_{\mu\nu}$, and $R^0{}_\alpha$ is still larger than the inhomogeneous Lorentz group.

A. Quantum mechanical operators and their Hilbert space (spherical case)

The energy H [(3.3) of I] is related to the momenta P [(3.4) of I] through the relation $H^2 = P^2 + M^2$. It indicates that the center of mass of the extended object moves freely. Therefore when we diagonalize H and P, the Hilbert space is a set of plane wave functions, namely, de Broglie waves.[14] Thus each sector of the Hilbert space can be written as

$$|K\rangle = e^{iK^\mu \xi_\mu}. \tag{8.1}$$

Since H is the total energy, the ξ_μ must be variables describing the center of the object, or equivalently the center of mass (in the case of spherical object, the center of mass coincides with the center of the object). In Sec. XI we will give unambiguous mathematical proof for this. The sectors $|K\rangle$ satisfy the quantum-mechanical equations

$$i\frac{\partial}{\partial \xi^\mu}|K\rangle = P_\mu |K\rangle, \tag{8.2}$$

and therefore K_μ are matrix elements of P_μ.[15] As a consequence, P_μ are the matrix representations of $i\partial/\partial\xi^\mu$ in the Heisenberg representation as well as in the Schrödinger representation.

Once the Hilbert space is thus fixed, the matrix representation of any quantum-mechanical operator is already determined. Any quantum-mechanical operator is a matrix representation of a function of ξ_μ, or more precisely, a function of ξ_μ and $\partial/\partial\xi^\mu$. Let us write the matrix element of a quantum-mechanical operator A between two sectors $\langle K_2|$ and $|K_1\rangle$ as $(K_2|A|K_1)$. The operator A is a matrix representation of some function, denoted by $\mathfrak{A}(\xi, \partial/\partial\xi)$. Then $(K_2|A|K_1)$ is determined to be

$$(K_2|A|K_1) = \int \langle K_2|\mathfrak{A}\left(\xi_\mu \frac{\partial}{\partial \xi^\mu}\right)|K_1\rangle d^3\xi. \tag{8.3}$$

Now we deal with matrix representations of $m_{\mu\nu}$ and Q_α. We first present some heuristic arguments which serve to determine these representations.

The $Q^0{}_\alpha$ are canonical conjugates of P_α, and therefore must be representations of ξ_α. Since $Q^0{}_\alpha$ are independent of time, they must be the Schrödinger representations of ξ_α. Consequently $Q_\alpha = Q^0{}_\alpha + \dot{Q}_\alpha t$ must be the canonical conjugates of P_α in the Heisenberg representation.

The $m_{\mu\nu}$ are total angular momentum operators. In fact only possible angular momenta of the spherical object are the orbital angular momenta (the spherical object does not fluctuate rotationally). Furthermore $m_{\mu\nu}$ and P_μ form an inhomogeneous Lorentz group. Then, since P_μ are the matrix representations of $i\partial/\partial\xi^\mu$, $m_{\mu\nu}$ must be the representations of $i(\xi_\mu\partial_\nu - \xi_\nu\partial_\mu)$.

The arguments given above for $Q^0{}_\alpha$ and $m_{\mu\nu}$, are only heuristic ones and need to be confirmed by mathematical proofs. We will do this in the following. We first assume that $m_{\mu\nu}$ are the representations of $i(\xi_\mu\partial_\nu - \xi_\nu\partial_\mu)$, and we will justify it at the end of this subsection. Then the matrix representations of N_α [(9.10) of I] should be

$$\begin{aligned}(K_2|N_\alpha|K_1) &= \frac{1}{M^2}\int \langle K^2|\partial^\nu(\xi_\alpha\partial_\nu - \xi_\nu\partial_\alpha)|K_1\rangle d^3\xi \\ &= \frac{1}{M^2}\int e^{-iK_2^\mu\xi_\mu}(\xi_\alpha\partial^\mu\partial_\mu - \partial^\mu\xi_\mu\partial_\alpha)e^{iK_1^\mu\xi_\mu}d^3\xi.\end{aligned} \tag{8.4}$$

The matrix representations of $\dot{Q}_\alpha N_0$ are a bit more complicated:

$$\begin{aligned}(K_2|\dot{Q}_\alpha N_0|K_1) &= \frac{1}{M^2}\int \left\langle K^2\left|\frac{\partial_\alpha}{\partial_0}\partial^\nu(\xi_0\partial_\nu - \xi_\nu\partial_0)\right|K_1\right\rangle d^3\xi \\ &= \frac{1}{M^2}\int e^{-iK_2^\mu\xi_\mu}\left(-M^2\xi_0\frac{\partial_\alpha}{\partial_0} - \partial^\nu\xi_\nu\partial_\alpha\right)e^{iK_1^\mu\xi_\mu}d^3\xi.\end{aligned} \tag{8.5}$$

In spite of their appearances, $\partial^\nu(\xi_\mu\partial_\nu - \xi_\nu\partial_\mu)$ are independent of t, as will be seen in the Sec. XI. Combining (8.4) and (8.5), we obtain the Heisenberg representations of $Q^0{}_\alpha$,

$$(K_2|Q^0{}_\alpha|K_1) = \int\left\langle K_2\left|\xi_\alpha - \xi_0\frac{\partial_\alpha}{\partial^0}\right|K_1\right\rangle d^3\xi. \tag{8.6}$$

The momenta ∂_μ are diagonal in our representation, and therefore the representations of $\partial_\alpha/\partial^0$ are P_α/H; that is $\dot{Q}_\alpha$. The ξ_0 is t by definition. Then the Heisenberg representations of variables ξ_α are obtained from (8.6) as

$$\int\langle K_2|\xi_\alpha|K_1\rangle d^3\xi = (K_2|Q^0{}_\alpha + \dot{Q}_\alpha t\,|K_1) = (K_2|Q_\alpha|K_1). \tag{8.7}$$

Thus, as expected, Q_α are now proven to be the Heisenberg representations of the center of the object (ξ_α signify the center, as will be shown in Sec. XI).

We can drop the second term in the right side of (8.6) in another way, namely, by transforming the Heisenberg representations (8.6) into the Schrödinger representations

$$(K_2|Q^0{}_\alpha|K_1)_{\rm Sch} = \int_{\rm Sch}\langle K_2|\xi_\alpha|K_1\rangle_{\rm Sch}d^3\xi. \tag{8.8}$$

Then $(K_2|Q^0{}_\alpha|K_1)_{\rm Sch}$ mean the Schrödinger representations of $Q^0{}_\alpha$. The $|K_1\rangle_{\rm Sch}$ indicate the time-independent ket vectors, namely, $\exp(i(k_x x + k_y y + k_z z))$.[14] Then the $Q^0{}_\alpha$ are the Schrödinger representations of the variables ξ_α. The heuristic argument on $Q^0{}_\alpha$ presented earlier in this section is now justified.

Throughout the above calculations, we have ignored a term proportional to ∂_α for the following reasons. Previously we have defined N_μ as $P^\nu m_{\nu\mu}/M^2$. However, we may define N_μ as $\frac{1}{2}(P^\mu m_{\nu\mu} + m_{\nu\mu}P^\nu)/M^2$. In the same way $Q^0{}_\alpha$ may be replaced by $N_\alpha - \frac{1}{2}(Q_\alpha N_0 + N_0\dot{Q}_\alpha)$. There are many other different ways of defining $Q^0{}_\alpha$ in connection with the ordering of the operators. However, the $Q^0{}_\alpha$'s thus defined differ among themselves only by terms proportional to ∂_α. In order to avoid getting involved with complexity which is not essential, we do not discuss this point further.

As a consequence of Eqs. (8.2), the P_μ are the matrix representations of $-i\partial_\mu = -i\partial/\partial\xi^\mu$ in the Heisenberg representation as well as in the Schrödinger representation. The matrix representations of ξ_α are obtained in (8.7) and (8.8) and are Q_α and $Q^0{}_\alpha$ in Heisenberg and Schrödinger representations. Since any quantum-mechanical operator is a matrix representation of some function of ξ_α and ∂_μ, we can always express it as a function of Q_α (or $Q^0{}_\alpha$) and P_μ. We illustrate this by choosing $m_{\mu\nu}$ as examples.

We first deal with $m_{0\alpha}$, namely,

$$(K_2|m_{0\alpha}|K_1) = i\int\langle K_2|\xi_0\partial_\alpha - \xi_\alpha\partial_0|K_1\rangle d^3\xi. \tag{8.9}$$

Since the representation of ξ_α are Q_α, we obtain the result

$$(K_2|m_{0\alpha}|K_1) = (K_2|\xi_0 P_\alpha - P_0 Q_\alpha|K_1), \tag{8.10}$$

where $\xi_0 = t$. The Q_α are $Q^0{}_\alpha + \dot{Q}_\alpha t$ and $P_0\dot{Q}_\alpha = -H\dot{Q}_\alpha = P_\alpha$. Finally the Heisenberg representations of $m_{0\alpha}$ are obtained,

$$(K_2|m_{0\alpha}|K_1) = (K_2|HQ^0{}_\alpha|K_1). \tag{8.11}$$

These expressions of $m_{0\alpha}$ are identical to the ones (4.3) of I (ignoring the ambiguity in the ordering).[16]

Earlier in this section the $m_{\alpha\beta}$ were assigned as the matrix representations of the operators $i(\xi_\alpha\partial_\beta - \xi_\beta\partial_\alpha)$. The matrix representations of $i\partial_\alpha$ are the quantum-mechanical momenta P_α describing fluctuation movement. According to (8.7), the representations of ξ_α are Q_α and signify the linear fluctuation of the center of the extended object about the origin of the coordinate system. Therefore $m_{\alpha\beta}$ of the spherical extended object are the quantum-mechanical orbital momenta due to the fluctuation of the extended object. Their matrix representations are

$$\begin{aligned}(K_2|m_{\alpha\beta}|K_1) &= -\int\langle K_2|\xi_\alpha\partial_\beta - \xi_\beta\partial_\alpha|K_1\rangle d^3\xi \\ &= Q_\beta P_\alpha - Q_\alpha P_\beta \\ &= Q^0{}_\beta P_\alpha - Q^0{}_\alpha P_\beta .\end{aligned} \tag{8.12}$$

The operators (8.12) commute with H. Consequently the Schrödinger representations of $m_{\alpha\beta}$ coincide with the Heisenberg representations and are also given by (8.12).

The results (8.11) and (8.12) agree with the expressions (4.3) and (4.4) of I for the relativistic angular momenta. Hence the heuristic argument on $m_{\mu\nu}$, given in the early part of this section, is now justified.

B. Quantum mechanical operators and their Hilbert space (nonspherical case)

Now we deal with the deformed extended object. The matrix representations for operators R^0, $\hat{S}$, $\hat{Q}^0$, and P can be easily constructed as it will be explained soon. Then, since any other quantum-mechanical operator can be expressed in terms of these operators, its matrix representation can be determined also.

The Hilbert space can be determined by considering P_μ and $\hat{S}_\alpha$ only, since only these two sets of operators appear in the four-momenta, especially in the energy (the Hilbert space is determined principally by the Hamiltonian). Now the P_μ commute with $\hat{S}_\alpha$. Consequently the Hilbert space is a product of two subspaces. It is not a direct product however, since P_μ include $\hat{S}^2$ through $\hat{M}$ ($\hat{M}$ can be written as a function of $\hat{S}^2$ instead of s, since s commutes with $\hat{S}_\alpha$. We will return to this point later). The representations of $\hat{S}_\alpha$ are determined solely by one of these two subspaces. We will occasionally call this subspace the Hilbert subspace for R_α and $\hat{S}_\alpha$, since not only the $\hat{S}_\alpha$, but also the essential part of the representations of R_α lie also in this subspace. The representations of $\hat{Q}^0{}_\alpha$ and P_μ depend only on the other subspace. We will occasionally call it the Hilbert subspace for the $\hat{Q}^0{}_\alpha$ and P_μ. Let us denote sectors in the Hilbert space as $|K,l,m\rangle$. Here l means the eigenvalue of $|\hat{S}|$. The meaning of the other symbols K and m will be explained later. Each of the sectors $|K,l,m\rangle$ is a product of two subsectors which we denote as $|K,l\rangle$ and $|l,m\rangle$; that is to say, $|K,l,m\rangle = |K,l\rangle|l,m\rangle$.

The sectors $|K,l,m\rangle$ have to satisfy, according to the law of quantum mechanics, the equation $-i\partial_t|K,l,m\rangle = H|K,l,m\rangle$, where H is a quantum-mechanical Hamiltonian (4.1). The way of dividing solutions $|K,l,m\rangle$ into $|K,l\rangle$ and $|l,m\rangle$ is not unique. Correspondingly there are numerous ways of separating this equation for $|K,l,m\rangle$ into two equations which $|l,m\rangle$ and $|K,l\rangle$ satisfy separately. In this paper we solve this equation by assuming $|l,m\rangle$ to be independent of time. We denote such $|l,m\rangle$ as $|l,m\rangle_{\rm Sch}$, where the suffix Sch is an abbreviation of Schrödinger. Then $|K,l\rangle$ alone satisfy the equation

$$-i\partial_t|K,l\rangle = \frac{l(l+1)/I + M(l)}{(1-\dot{Q}^2)^{1/2}}|K,l\rangle.$$

Here H is diagonal in both $|K,l\rangle$ and $|l,m\rangle_{\rm Sch}$, and $\{l(l+1)/I + M(l)\}/(1-\dot{Q}^2)^{1/2}$ are its diagonal elements in $|l,m\rangle_{\rm Sch}$.

The representations $\hat{S}_\alpha$ are determined solely in the subspace $|l,m\rangle_{\rm Sch}$, as is mentioned previously. However, the matrices $R^0{}_\alpha$ depend on both of $|K,l\rangle$ and $|l,m\rangle_{\rm Sch}$. Still, as will become clear in the end of this section, we can rewrite $R^0{}_\alpha$ as products $R^{0'}{}_\alpha R^{2''}{}_\alpha$, where $R^{0'}{}_\alpha$ and $R^{0''}{}_\alpha$ depend separately on $|K,l\rangle$ and $|l,m\rangle_{\rm Sch}$, respectively.

Then the matrices $R^{0''}{}_\alpha$ and $\hat{S}_\alpha$ are completely independent of the matrices P_μ and therefore of $\dot{Q}_\alpha$. Therefore, let us replace $\dot{Q}$ by 0 in the solution (3.1). Then the expression (3.1) of the deformed object ϕ indicates that operators $R^{0''}{}_\alpha$ and S_α are the angles and angular velocities in the frame of reference where the center of the object is at rest (in such a frame of reference the center of mass coincides with the center). Correspondingly we construct the matrix representations of $R^{0''}{}_\alpha$ and $\hat{S}_\alpha$ as the angle operators and angular momentum operators of the object rotating around the origin.

Once matrix representations of $\hat{S}_\alpha$ are obtained, the representations of the rest of the operators P_μ and $\hat{Q}^0$ can be constructed exactly in the same way as in the spherical case. Finally, the matrices $R^{0'}$ can be constructed using the Hilbert subspace of P_μ, and $\hat{Q}^0$. Therefore, in principle, we should construct the matrices $R^{0''}{}_\alpha$ and S_α first, and then construct the P_μ and $\hat{Q}^0$ later. However, the procedure of constructing the matrices $R^{0''}{}_\alpha$ and $\hat{S}_\alpha$ is lengthy, and it is preferable to deal with them later. We first construct the matrix representations for $\hat{Q}^0$ and P in the following.

The Hamiltonian H is related to momenta P_α through the relation $H = \hat{M} + P^\alpha P_\alpha$, where $\hat{M}$ is a function of s only. Since s commutes with P and $\hat{Q}^0$, $\hat{M}$ is simply a C-number in this Hilbert subspace. Furthermore, $\hat{Q}^0$ and P satisfy the canonical commutation relation just as Q^0 and P of the spherical object do [see (7.4)]. Then the results obtained in the previous Subsec. A are all valid here if we replace Q^0 in Subsec. A by $\hat{Q}^0$. First of all, each sector of the Hilbert subspace concerned is given as

$$|K,l\rangle = \exp(iK^\mu\xi_\mu) \tag{8.13}$$

in the Heisenberg representations, and

$$|K,l\rangle_{\rm Sch} = \exp(i(K_x\xi_x + K_y\xi_y + K_z\xi_z)) \tag{8.14}$$

in the Schrödinger representations, where $K^\mu K_\mu = M^2 = l(l+1) + M(l)$. Secondly, the $\hat{Q}^0{}_\alpha$ are the matrix representations of the coordinates ξ_α in the Schrödinger representation, namely,

$$\begin{aligned}(K_2,l|\hat{Q}^0{}_\alpha|K_1,l)_{\rm Sch} &= (K_2,l|\hat{N}_\alpha - \dot{Q}_\alpha\hat{N}_0|K_1,l)_{\rm Sch}\\ &= \int\langle K_2,l|\xi_\alpha|K_1,l\rangle_{\rm Sch}\,d^3\xi.\end{aligned} \tag{8.15}$$

Then the $\hat{Q}^0{}_\alpha + \dot{Q}t$, denoted as $\hat{Q}_\alpha$, are the Heisenberg representations of ξ_α; that is,

$$\begin{aligned}\int\langle K_2,l|\xi_\alpha|K_1,l\rangle d^3\xi &= (K_2,l|\hat{Q}_\alpha|K_1,l).\end{aligned} \tag{8.16}$$

Finally, $\hat{m}_{\mu\nu}$ are the matrix representations of $-i(\xi_\mu\partial_\nu - \xi_\nu\partial_\mu)$,

$$\begin{aligned}(K_2,l|\hat{m}_{\mu\nu}|K_1,l) &= -i\int\langle K_2,l|\xi_\mu\partial_\nu - \xi_\nu\partial_\mu|K_1,l\rangle d^3\xi.\end{aligned} \tag{8.17}$$

Now we deal with R^0 and $\hat{S}$. These two kinds of operators do not appear in the existing quantum mechanics which was initially obtained from the classical mechanics of a point particle through the quantization procedure. In the preceding sections we have seen where these new operators $\hat{S}$ and P originate from. The relations of $\hat{S}_\alpha$ to other quantum operators P_μ and $m_{\mu\nu}$ are already clarified [see (6.5)], and $\hat{S}_\alpha$ are found to be representations of infinitesimal rotation operators [see (6.12)]. However, R^0 have appeared so far only in the function (3.2) and have not been integrated properly in the new quantum mechanics. Therefore in order to determine the matrix representations of R^0 and $\hat{S}$, we have to start from the solution (3.2).

The wave function (3.2), which we have adopted for the vacuum value of ψ, describes an object moving with the velocity $\dot{Q}_\alpha = \tanh A_\alpha$, and rotating with the angular velocity S around its center. Replacing N_μ and A_α by zeros (equivalently $\dot{Q}_\alpha$ and P_μ by zeros) in (3.2), we obtain the function

$$\exp(-(R^0{}_\alpha + S_\alpha t)(x_\beta\partial_\gamma - x_\gamma\partial_\beta))\rho(x,y,z,s). \tag{8.18}$$

This is an object simply rotating with the angular velocity $\vec{S} = (S_x, S_y, S_z)$. The angular momenta of this object can be obtained by replacing $\dot{Q}_\alpha$ by zeros in m_{xy} [(5.2)], or equivalently, by replacing P_α by zeros in the $m_{\beta\gamma}$ [(7.6)]. They are $\hat{S}_\alpha$ and satisfy the commutation relations (6.12). Consequently $\hat{S}_\alpha$ are the matrix representations of three infinitesimal rotation operators for the object (8.18). Then we determine the matrices R in such a way that the function (8.18) is left invariant for the infinitesimal rotation $(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + i\hat{S}_\alpha$. Now the R^0 are the Schrödinger representations (approximative) of the matrices $R = R^0 + St$, as will be proven later. Then the following function is the Schrödinger representation of the function (8.18),

$$\exp(-R^0{}_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))\rho(x,y,z,s). \tag{8.19}$$

This function is easier to deal with than the function (8.18). We first determine R^0 in the Schrödinger representation so that the (8.19) is invariant for the rotations $(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + i\hat{S}_\alpha$. The function (8.19) expresses a rotated object and

$R^0{}_x$, $R^0{}_y$, and $R^0{}_z$ are matrix representations of angles of rotations around x, y, and z axes, respectively (we ignore the presence of s in ρ, since s commutes with $\hat{S}_\alpha$. We will come back to this point in Sec. IX). We will denote these angles by φ_α. Later, in Sec. XI, we will show in detail the precise mathematical definition of φ_α [see (11.8)].

The angles φ_α determine the orientations of the rotated object (8.19). We define the orientation of the object (8.19) by choosing three unit vectors which are fixed on this object and are orthogonal mutually. We denote these three vectors by

$$(\tau_x^{(1)},\tau_y^{(1)},\tau_z^{(1)}),(\tau_x^{(2)},\tau_y^{(2)},\tau_z^{(2)}),\text{ and }(\tau_x^{(3)},\tau_y^{(3)},\tau_z^{(3)}). \tag{8.20}$$

For the sake of convenience we choose these three vectors to lie on the axes x, y, and z, respectively, when $\varphi_\alpha = 0$. In other words, the three fixed vectors of the unrotated object $\rho(x, y, z, s)$ lie on the axes x, y, and z. Naturally they rotate as the object rotates. Then the three vectors (8.20) satisfy the relations

$$e^{\varphi(\tau\times\partial)}\tau_\alpha^{(i)} = \delta_{i\alpha}, \tag{8.21}$$

where

$$\varphi(\tau\times\partial) = \varphi_x\sum_{(i)}(\tau_y^{(i)}\partial_{\tau_z^{(i)}} - \tau_z^{(i)}\partial_{\tau_y^{(i)}}) + \text{cyclic permutation on } x, y, \text{ and } z. \tag{8.22}$$

The $\delta_{i\alpha}$ are defined as

$$\delta_{1x} = \delta_{2y} = \delta_{3z} = 1,$$

and all other $\delta_{i\alpha}$ are zero.

Although there appear nine components $\tau^i_{(\alpha)}$ in the relations (8.21), only three of them are independent. Since the number of components φ are also three, we can determine the φ as functions of τ,

$$\varphi_\alpha = \varphi_\alpha(\tau). \tag{8.23}$$

Now we deal with the quantum-mechanical operators $\hat{S}$. Let us denote the projection of the point (x, y, z) on three vectors (8.20) by $\tau^{(1)}$, $\tau^{(2)}$, and $\tau^{(3)}$. In other words, we locate each point of the space by the coordinate system fixed on the extended object. Then the infinitesimal rotational transformation of this coordinate system $(\tau^{(1)}, \tau^{(2)}, \tau^{(3)})$ is realized by the operators

$$\tau^{(i)}\partial_{\tau^{(j)}} - \tau^{(j)}\partial_{\tau^{(i)}}. \tag{8.24}$$

Since this coordinate system $(\tau^{(1)}, \tau^{(2)}, \tau^{(3)})$ is fixed on the extended object, these operators (8.24) rotate the extended object. Now the extended object (8.19) has to conserve the operators $(x_\beta\partial_\gamma - x_\gamma\partial_\alpha) + i\hat{S}_\alpha$. It means that the coordinate system (x, y, z) and the extended object are both rotated simultaneously and the orientation of the object relative to the coordinate system (x, y, z) remains unaltered. Then, when $\varphi_\alpha = 0$, namely, when the body-fixed coordinate system $(\tau^{(1)}, \tau^{(2)}, \tau^{(3)})$ coincides with the coordinate system (x, y, z), the $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ have to be the matrix representations of the following operators, respectively,

$$i(\tau^{(2)}\partial_{\tau^{(3)}} - \tau^{(3)}\partial_{\tau^{(2)}}),\quad i(\tau^{(3)}\partial_{\tau^{(1)}} - \tau^{(1)}\partial_{\tau^{(3)}}),$$
$$i(\tau^{(1)}\partial_{\tau^{(2)}} - \tau^{(2)}\partial_{\tau^{(1)}}). \tag{8.25}$$

When $\varphi_\alpha \neq 0$, the following three directions coincide with the axes x, y, and z.

$$e^{\varphi(\tau\times\partial)}\tau^{(i)}e^{-\varphi(\tau\times\partial)}, \tag{8.26}$$

where $i = 1$, 2, and 3, respectively. Therefore we rotate $\tau^{(i)}$ such that these three directions (8.26) are rotated in the same way as three axes x, y, and z are rotated by the operators $(x_\beta\partial_\gamma - x_\gamma\partial_\beta)$. Then in order for $(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + iS_\alpha$ to vanish, the $\hat{S}_\alpha$ have to be representations of the following operators denoted as $-i(\mathring{\tau}\times\partial)$,

$$-i(\mathring{\tau}\times\partial)_\alpha = ie^{-\varphi(\tau\times\partial)}(\tau^{(i)}\partial_{\tau^{(j)}} - \tau^{(j)}\partial_{\tau^{(i)}})e^{\varphi(\tau\times\partial)}. \tag{8.27}$$

Here (i, j) is (2.3) or (3.1) or (1.2) according to whether $\alpha = x$ or y or z.

The commutation relations between the operators (8.27) and φ are rather complicated. Correspondingly the commutation relations between quantum rotation operators $\hat{S}_\alpha$ and quantum R_α are not simple. However, there exists one simple commutation relation, namely, the canonical relation between a specific component of the operators (8.27) and $|\varphi| = (\varphi^2{}_x + \varphi^2{}_y + \varphi^2{}_z)^{1/2}$. Let us denote the component of the vectors (8.27) along the direction $(\varphi_x, \varphi_y, \varphi_z)$ by $-i(\mathring{\tau}\times\partial)_\varphi$, namely,

$$-i\sum_\alpha\frac{\varphi_\alpha}{|\varphi|}(\mathring{\tau}\times\partial)_\alpha = -i(\mathring{\tau}\times\partial)_\varphi. \tag{8.28}$$

Then $|\varphi|$ and $-i(\mathring{\tau}\times\partial)_\varphi$ satisfy the canonical commutation relation

$$[(\mathring{\tau}\times\partial)_\varphi, |\varphi|] = 1. \tag{8.29}$$

Now we will determine the Hilbert subspace for R^0 and $\hat{S}$. The integrated Hamiltonian of the function (8.18) is $\hat{M}$ (4.5). Then each sector of the Hilbert subspace, which we denote as $|\ \rangle$, has to satisfy the equation

$$i\partial_t|\ \rangle = \hat{M}|\ \rangle. \tag{8.30}$$

We first deal with it in the Schrödinger representation, which we denote as $|\ \rangle_{\text{Sch}}$. Then $|\ \rangle_{\text{Sch}}$ satisfies the Schrödinger equation

$$E|\ \rangle_{\text{Sch}} = \hat{M}|\ \rangle_{\text{Sch}}. \tag{8.31}$$

The integrated Hamiltonian $\hat{M}$ consists of $\hat{S}^2/I$ and $M(s)$ as is seen in (4.5). The I and $M(s)$ are given in (4.3) and (4.4), and are functions of s through ρ. Now s is a function of $|\hat{S}|$ through the relation $|\hat{S}| = sI$, and consequently $\hat{M}$ is a function of $|\hat{S}|$ only, as is mentioned previously in the beginning of this subsection. Finally, each section $|\ \rangle_{\text{Sch}}$ in an eigenfunction of $|\hat{S}|$. The Hilbert subspace is then independent of the length τ^2 and depends on the angular variables in τ space only. This situation arises because the $x^\mu\partial_\mu\phi$ is not the zero mode and no quantum number appears in relation to τ^2.

To determine the Hilbert space explicitly, we first rewrite $\hat{M}$ as a function of $|\hat{S}|$. We denote the result as $\hat{M}(|\hat{S}|)$. Then we replace $|\hat{S}|$ by the operator $(\Sigma_\alpha\{i(\mathring{\tau}\times\partial)_\alpha\}^2)^{1/2}$. Then each sector $|\ \rangle_{\text{Sch}}$ satisfies the differential equation

$$E|\ \rangle = \hat{M}\left(\left(\sum_\alpha\{i(\mathring{\tau}\times\partial)_\alpha\}^2\right)^{1/2}\right)|\ \rangle_{\text{Sch}}. \tag{8.32}$$

The solutions of this equation are eigenfunctions of the operator $(\Sigma_\alpha\{i(\mathring{\tau}\times\partial)_\alpha\}^2)^{1/2}$ and are the set of spherical harmonics denoted customarily as $Y_{l,m}$. Each sector of the Hilbert subspace can be distinguished by two numbers l and m as $|l,m\rangle_{\text{Sch}}$, namely,

$$|l,m\rangle_{\rm Sch} = Y_{l,m}(\theta_\tau,\phi_\tau). \tag{8.33}$$

Now we are ready to determine matrix representations $R^{0\prime\prime}$ and $\hat{S}$. In fixing the Hilbert subspace (8.33), we have started from the function (8.18) instead of (3.2); that is, we have replaced $\dot{Q}$ by zeros. Now the matrices R^0 appearing in the function (3.2) can be written as products $R^{0\prime}R^{0\prime\prime}$. The matrices $R^{0\prime}$ in the subspace $|K,l\rangle$ should be replaced by unities, as $\dot{Q}$ vanish. The representations of φ, which we obtain in the Hilbert subspace $|l,m\rangle$ [(8.33)], are independent of $\dot{Q}$, and therefore are matrices $R^{0\prime\prime}$. In conclusion, the matrix representations $R^{0\prime\prime}$ and $\hat{S}$ are determined to be

$$(l_2m_2|R^{0\prime\prime}{}_\alpha|l_1m_1) = \int \langle l_2m_2|\varphi_\alpha(\tau)|l_1m_1\rangle_{\rm Sch}\, d\Omega$$

$$= \int Y_{l_2m_2}(\theta_\tau\phi_\tau)\varphi_\alpha(\tau)Y_{l_1m_1}(\theta_\tau\phi_\tau)\sin\theta_\tau\, d\theta_\tau\, d\phi_\tau, \tag{8.34}$$

$$\begin{aligned}(l_2m_2|\hat{S}_\alpha|l_1m_1) = \int \langle l_2m_2|e^{-\varphi(\tau\times\partial)}(\tau^{(i)}\partial_{\tau^{(j)}} \\ - \tau^{(j)}\partial_{\tau^{(i)}})e^{\varphi(\tau\times\partial)}|l_1m_1\rangle d\Omega \\ = \int Y_{l_2m_2}(\theta_\tau\phi_\tau)e^{-\varphi(\tau\times\partial)}(\tau^{(i)}\partial_{\tau^{(j)}} \\ - \tau^{(j)}\partial_{\tau^{(i)}})e^{\varphi(\tau\times\partial)}Y_{l_1m_1}(\theta_\tau\phi_\tau) \\ \times\sin\theta_\tau\, d\theta_\tau\, d\phi_\tau.\end{aligned} \tag{8.35}$$

With $\hat{S}$ and R^0 thus determined, the infinitesimal operators $(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + i\hat{S}_\alpha$ are conserved by the function (8.19), namely,

$$\begin{aligned}&[(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + i\hat{S}_\alpha, \\ &\quad \exp(-R^0{}_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))\rho(x,y,z,s)] = 0.\end{aligned} \tag{8.36}$$

Here s is proportional to $|\hat{S}| = (\hat{S}^2{}_x + \hat{S}^2{}_y + \hat{S}^2{}_z)^{1/2}$ and commutes with $\hat{S}_\alpha$. In fact the $\hat{S}$ and R^0 satisfy, by definition, following stronger relations,

$$\begin{aligned}&[x_\beta\partial_\gamma - x_\gamma\partial_\beta + i\hat{S}_\alpha, \\ &\quad \exp(-R^0{}_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))x_\sigma] = 0.\end{aligned} \tag{8.37}$$

This is because the $\exp(-\varphi_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))x_\sigma$ are three coordinates relative to the axes $(\tau^{(1)}, \tau^{(2)}, \tau^{(3)})$ fixed on the body, and therefore are not altered when the space (x, y, z) and the body are rotated simultaneously.

The commutation relations between the matrices $R^0{}_\alpha$ and the matrices $\hat{S}_\alpha$ seem to be rather complicated. However, there is one simple relationship between these two sets of matrices. In the Hilbert subspace introduced above, the commutation relation (8.29) can be expressed as

$$[\hat{S}_R, |R^0|] = i, \tag{8.38}$$

where $|R^0| = (R^{02}{}_x + R^{02}{}_y + R^{02}{}_z)^{1/2}$. At each matrix element, the vector R^0 has some direction and $\hat{S}_R$ is the component of the vector $\hat{S}$ along the direction of R^0.

Now the vector $\hat{S}$ also has some direction at each matrix element. Then, from the relation (8.29) we can easily prove that the direction of $\hat{S}$ coincides with the direction of R^0.[17] Finally we obtain the relation

$$[|\hat{S}|, |R^0|] = i. \tag{8.39}$$

Now we transform these matrices (8.34) into the Heisenberg representation. The integrated Hamiltonian of the function (8.18) is $\hat{M}$ [(4.5)], and satisfies the canonical commutation relation

$$[\hat{M}, |R^0|] = -is. \tag{8.40}$$

This commutation relation can be obtained from the relations (4.20) and (8.39). The vector R^0 is parallel to $\hat{S}$ as mentioned previously, and therefore the following relations are valid [see (8.40)],

$$[\hat{M}, R^0{}_\alpha] = -iS_\alpha. \tag{8.41}$$

The $\hat{M}$ [(4.5)] is the Hamiltonian (4.1) with vanishing $\dot{Q}$, namely, the Hamiltonian in the approximation where the translational fluctuation energy is ignored (the energy due to rotational fluctuation is still included). Then $R_\alpha = R^0{}_\alpha + S_\alpha t$ are Heisenberg representations of $\varphi_\alpha(\tau)$, and therefore are Heisenberg operators in this approximation, namely,

$$e^{i\hat{M}t}R^0{}_\alpha e^{-i\hat{M}t} = R^0{}_\alpha + S_\alpha t = R_\alpha. \tag{8.42}$$

Now s is a function of $|\hat{S}|$ and vice versa, as is mentioned previously. The I, which is a function of s, is a function of $|\hat{S}|$. Consequently, each of $S_\alpha = \hat{S}_\alpha/I$ is a function of $\hat{S}_\alpha$'s, namely, $S_\alpha = F_\alpha(\hat{S}_\alpha$'s). Finally, from (8.42), we obtain the matrix representations of R as follows,

$$\begin{aligned}(l_2m_2|R_\alpha|l_1m_1) &= \int \langle l_2m_2|\varphi_\alpha(\tau)|l_1m_1\rangle d\Omega \\ &= (8.34) + F_\alpha((8.35)\text{'s})\times t.\end{aligned} \tag{8.43}$$

The Heisenberg representations of $R^0{}_\alpha$ without approximation can be obtained by transforming it by e^{iHt} instead of $e^{i\hat{M}t}$. First,

$$[H, R^0{}_\alpha] = -iS_\alpha/(1-\dot{Q}^2)^{1/2} \tag{8.44}$$

and, therefore,

$$e^{iHt}R^0{}_\alpha e^{-iHt} = R^0{}_\alpha + (S_\alpha/(1-\dot{Q}^2)^{1/2})t. \tag{8.45}$$

Finally, by transforming the relations (8.36) and (8.37) into the ones in the approximate Heisenberg representation by the canonical transformation operator $e^{i\hat{M}t}$, we obtained the following conservation relations which we have been longing for:

$$\begin{aligned}&[(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + i\hat{S}_\alpha, \\ &\quad \exp(-(R^0 + St)_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))\rho(x,y,z,s)] = 0,\end{aligned} \tag{8.46}$$

$$\begin{aligned}&[(x_\beta\partial_\gamma - x_\gamma\partial_\beta) + i\hat{S}_\alpha, \\ &\quad \exp(-(R^0 + St)_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))x_\sigma] = 0.\end{aligned} \tag{8.47}$$

Now we briefly deal with the representations in the total Hilbert space. The sectors in this entire space are the products of $|K,l\rangle$ and $|l,m\rangle_{\rm Sch}$, and satisfy the following fundamental equation of quantum mechanics,

$$i\partial_t|K,l\rangle|l,m\rangle_{\rm Sch} = H|K,l\rangle|l,m\rangle_{\rm Sch}. \tag{8.48}$$

In our representation $\hat{S}^2$ is diagonal and is $l(l+1)$, and therefore Eq. (8.48) becomes

$$i\partial_t|K,l\rangle|l,m\rangle_{\rm Sch} = \frac{l(l+1)/I + M(l)}{(1-\dot{Q}^2)^{1/2}}|Kl\rangle|l,m\rangle_{\rm Sch}. \tag{8.49}$$

Since $|l,m\rangle_{\rm Sch}$ are independent of time, the subsectors $|K,l\rangle$ alone satisfy Eq. (8.49), as is mentioned in the beginning of

this Subsec. B. The representations of P, $\hat{Q}^0$, and $\hat{S}$ in this entire Hilbert space, can be obtained by multiplying K_μ and the matrices (8.15) and (8.35) by unit matrices either in $|K,l\rangle$ space or in $|l,m\rangle$ space. The complete representations of R^0 can be obtained by multiplying $R^{0\prime}$ by $R^{0\prime\prime}$ [(8.34)], namely,

$$(K_2 l_2 m_2|R^0{}_\alpha|K_1 l_1 m_1)$$
$$= \int \langle l_1 m_2|\langle K_2 l_2|\varphi_\alpha|K_1 l_1\rangle|l_1 m_1\rangle d\Omega\, d^3\xi$$
$$= (K_2 l_2|R^{0\prime}{}_\alpha|K_1 l_1)(l_2 m_2|R^{0\prime\prime}{}_\alpha|l_1 m_1). \tag{8.50}$$

Here the matrices $R^{0\prime}{}_\alpha$ are independent of α and are

$$(K_2 l_2|R^{0\prime}{}_\alpha|K_1 l_1) = \int \langle K_2 l_2|1|K_1 l_1\rangle d^3\xi. \tag{8.51}$$

The presence of the factors $R^{0\prime}$ in R^0 does not alter the commutation relations between R^0 and $\hat{S}$. This is because the $|K,l\rangle$ [(8.13)] depend on $\hat{S}$ through $\hat{S}^2$ and therefore commute with $\hat{S}$. Consequently, all the relations from (8.39) till (8.45) are still valid for the full representations of R^0 (namely $R^{0\prime}R^{0\prime\prime}$).

Before we end the discussions on the R^0 and $\hat{S}$, we wish to make a remark on the intrinsic spin. The matrices R^0 are not diagonal at all with respect to the quantum number l. Then the Hilbert space cannot be an irreducible one in connection with the intrinsic angular momenta $\hat{S}$. Furthermore, the $|R^0|$ and $|\hat{S}|$ satisfy the canonical commutation relation (8.39). Hence the Hilbert subspace for $\hat{S}$ and $R^{0\prime\prime}$ is necessarily a complete set of spherical harmonics and our fluctuating object (3.1) cannot have a definite intrinsic spin.

Now we have prepared sufficiently to determine the representations of $m_{\mu\nu}$ and N_μ. The angular momenta $m_{\alpha\beta}$ consist of the orbital angular moment $\hat{m}_{\alpha\beta}$ and the intrinsic angular momenta $\hat{S}$. The time-spatial parts $m_{0\alpha}$ should consist of $\xi m_{0\alpha}$ and the contribution from $\hat{S}$. Then relativistic center of mass N_μ also should consist of $\hat{N}_\mu$ and the contribution from $\hat{S}$.[18] Hence, the matrix representations of these operators $m_{\mu\nu}$ and N_μ will be determined as we express them in terms of $\hat{m}_{\mu\nu}$, $\hat{N}_\mu$, and $\hat{S}$. We will do this in the following.

From the assignment (7.10) and the relations (7.3), we can rewrite N_μ in terms of $\hat{N}_\mu$ and $\hat{S}$ as

$$N_0 = \hat{N}_0,$$
$$N_\alpha = \hat{N}_\alpha + \left(-1 + \frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{[P\times\hat{S}]_{\beta\gamma}}{P^2}, \tag{8.52}$$

where $P^2 = P^2{}_x + P^2{}_y + P^2{}_z$. Inserting the expression (8.52) into $m_{\mu\nu}$, namely, (5.1) and (5.2), and using the expressions (7.7) and (7.8) for $\hat{m}_{\mu\nu}$, we obtain the results

$$m_{0\alpha} = \hat{m}_{0\alpha} - \left\{-1 + \frac{1}{(1-\dot{Q}^2)^{1/2}}\right\} \times \frac{(1-\dot{Q}^2)^{1/2}[\dot{Q}\times\hat{S}]_{\beta\gamma}}{\dot{Q}^2}, \tag{8.53}$$

$$m_{\alpha\beta} = \hat{m}_{\alpha\beta} + \hat{S}_\gamma. \tag{8.54}$$

These two relations are consistent with the expressions (7.5) and (7.6) for $m_{\mu\nu}$.

Matrix representations of $\hat{m}_{\mu\nu}$ and $\hat{S}_\alpha$ are already obtained in (8.17) and (8.34). The $\dot{Q}_\alpha$ are diagonal and are K_α/K^0 [see (8.13)]. In the expressions (8.53) and (8.54), the $m_{\mu\nu}$ are given in terms of the matrices $\hat{m}_{\mu\nu}$, $\hat{S}_\alpha$, and $\dot{Q}_\alpha$. Thus the matrix representations of $m_{\mu\nu}$ are now determined.

IX. LORENTZ COVARIANCE (2)

In this section we will accomplish our ultimate purpose; that is to say, we will complete the proof of the Lorentz covariance of the solution (3.1). Everything undertaken so far was focused on this purpose. All the quantities appearing in the solution (3.1) are properly defined, and necessary formulas are fully derived already. Now all that remains is to combine these formulas in order to prove the covariance.

We first note that N is a Lorentz vector and therefore $N^\mu\partial_\mu$ is covariant. Therefore it is sufficient if we prove the covariance of the function

$$\exp(-A^\alpha(t\partial_\alpha + x_\alpha\partial_t))$$
$$\times\exp(-(R^0 + St)_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))\rho(x,y,z,s)$$
$$= \exp(-(R^0 + ST)_\alpha(\hat{X}_\beta\partial_\gamma - \hat{X}_\gamma\partial_\beta))\rho(\hat{X},\hat{Y},\hat{Z},s)$$
$$= \rho(\hat{X}',\hat{Y}',\hat{Z}',s), \tag{9.1}$$

where

$$\hat{X}_\sigma = \exp(-A^\alpha(t\partial_\alpha + x_\alpha\partial_t))x_\sigma \tag{9.2}$$

and

$$\hat{X}'_\sigma = \exp(-A^\alpha(t\partial_\alpha + x_\alpha\partial_t)) \times\exp(-(R^0 + St)_\alpha(x_\beta\partial_\gamma - x_\gamma\partial_\beta))x_\sigma$$
$$= \exp(-(R^0 + ST)_\alpha(\hat{X}_\beta\partial_\gamma - \hat{X}_\gamma\partial_\beta))\hat{X}_\sigma. \tag{9.3}$$

Since s is proportional to $\hat{S}$ and therefore commutes with $\hat{S}_\alpha$, it is sufficient if we prove the Lorentz covariance of $\hat{X}'$, namely,[19]

$$[(t\partial_x + x\partial_t) + im_{0x},\hat{X}'_\sigma] = 0. \tag{9.4}$$

We divide $m_{0\alpha}$ [(8.53)] into two parts,

$$m_{0\alpha} = m^{(1)}_{0\alpha} + m^{(2)}_{0\alpha}, \tag{9.5}$$

where

$$m^{(1)}_{0\alpha} = \hat{m}_{0\alpha} = P_0\hat{Q}^0{}_\alpha \tag{9.6}$$

and

$$m^{(2)}_{0\alpha} = -\left\{-1 + \frac{1}{(1-\dot{Q}^2)^{1/2}}\right\} \times\frac{(1-\dot{Q}^2)^{1/2}[\dot{Q}\times\hat{S}]_{\beta\gamma}}{\dot{Q}^2}. \tag{9.7}$$

The $\dot{Q}$ commute with $m^{(2)}_{0\alpha}$ and therefore the Lorentz transformation properties of $\dot{Q}$ are determined by $m^{(1)}_{0\alpha}$ only. Since the commutation relations between $\dot{Q}$ and $m^{(1)}_{0\alpha}$ are identical to ones in the case of a spherical object, the Lorentz transformation properties of $\dot{Q}$ derived in Sec. V of the previous paper I are also valid in the present case. The same is true for $\hat{Q}^0$ (note that $\dot{Q}$ and Q^0 are written as $-\dot{Q}^0$ and $-Q^0$ in I). As a result, the formulas for the infinitesimal transformation of $\hat{X}'$ derived in the previous paper [(13.7) of I] are still valid here, namely,

$$\epsilon[(x\partial_t + t\partial_x) + im^{(1)}_{0x},\hat{X}_\sigma] = [\Omega\times\hat{X}]_\sigma, \tag{9.8}$$

where

$$\Omega = \frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\left(-1 + \frac{1}{(1-\dot{Q}^2)^{1/2}}\right)[\dot{Q}\times\theta].$$

The θ is an angular vector and is $(\epsilon,0,0)$. The relations (9.8) can be rewritten as

$$[(x\partial_t + t\partial_x) + im^{(1)}_{0x},\hat{X}_\sigma] = \frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right) \times [\dot{Q}\times[\hat{X}\times\partial]]\hat{X}_\sigma, \tag{9.9}$$

where

$$[\hat{X}\times\partial]_\alpha = \hat{X}_\beta \frac{\partial}{\partial \hat{X}_\gamma} - \hat{X}_\gamma \frac{\partial}{\partial \hat{X}_\beta}.$$

Adding $m^{(2)}_{0x}$ to $m^{(1)}_{0x}$ in (9.9), we obtain the relation

$$[(x\partial_t + t\partial_x) + im_{0x},\hat{X}'_\sigma] = \frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right) \times\{[\dot{Q}\times[\hat{X}\times\partial]] + i[\dot{Q}\times\hat{S}]\} \times\exp(-(R^0+ST)_\alpha(\hat{X}_\beta\partial_\gamma - \hat{X}_\gamma\partial_\beta))\hat{X}_\sigma. \tag{9.10}$$

However, this vanishes because of the conservation relation (8.47). Thus the covariance of $\hat{X}'$, namely (9.4), is now proven.

Now we will prove the rotational covariance of $\hat{X}'$, namely,

$$[(x\partial_y - y\partial_x) + im_{xy},\hat{X}'_\sigma] = 0. \tag{9.11}$$

We divide the m_{xy} [(8.54)] into two parts,

$$m_{xy} = m^{(1)}_{xy} + m^{(2)}_{xy}, \tag{9.12}$$

$$m^{(1)}_{xy} = \hat{S}_z, \quad m^{(2)}_{xy} = \hat{m}_{xy} = [\hat{Q}^0\times P]_{xy}. \tag{9.13}$$

The operator $x\partial_y - y\partial_x + im^{(2)}_{xy}$ simply rotates $\hat{X}$ and can be written as $\hat{X}\partial_{\hat{Y}} - \hat{Y}\partial_{\hat{X}}$, namely,

$$[x\partial_y - y\partial_x + im^{(2)}_{xy},\hat{X}_\sigma] = [\hat{X}\partial_{\hat{Y}} - \hat{Y}\partial_{\hat{X}},\hat{X}_\sigma]. \tag{9.14}$$

Adding $m^{(1)}_{xy}$ to $m^{(2)}_{xy}$ in (9.14) we obtain the relation

$$[(x\partial_y - y\partial_x) + im_{xy},\hat{X}'_\sigma] = [\hat{X}\partial_{\hat{Y}} - \hat{Y}\partial_{\hat{X}} + \hat{S}_z, \exp(-(R^0+ST)_\alpha(\hat{X}_\beta\partial_\gamma - \hat{X}_\gamma\partial_\beta))\hat{X}_\sigma]. \tag{9.15}$$

This vanishes because of the conservation relation (8.47). The proof of the Lorentz covariance of the extended object, including spatial rotation, is now accomplished.

X. POINCARÉ COVARIANCE

It still remains to prove the Poincaré covariance of the solution ϕ [(3.1)], namely,

$$[\partial_\mu + iP_\mu,\phi] = 0. \tag{10.1}$$

The ϕ can be expressed as a function of $\dot{Q}_\alpha$, Q_α, and R_α, apart from the coordinates x, y, and z. The commutation relations between P_μ and Q_α, and also between P_0 and $R^0{}_\alpha$, are known already [see (8.44)]. The commutation relations between P_α and R_α can be obtained from the relations (8.41); that is,

$$[P_\alpha,R_\beta] = [P_\alpha,R^0{}_\beta] = -i\dot{Q}_\beta S_\alpha/(1-\dot{Q}^2)^{1/2}. \tag{10.2}$$

Then we can confirm relations (10.1) by calculating $[P_\mu,\phi]$ by using these commutation relations. However, there is another way of explaining clearly how our solution (3.1) is Poincaré covariant.

The solution ϕ [(3.1)] depends on t only through the operators $Q_\alpha = Q^0{}_\alpha + \dot{Q}t$ and $R^0{}_\alpha + S_\alpha T = R^0{}_\alpha + S_\alpha\{t-(\dot{Q},x)\}/(1-\dot{Q}^2)^{1/2}$. When one transforms the solution ϕ into the Schrödinger representation, denoted as $\phi_{\rm Sch}$, the operators Q_α and $R^0{}_\alpha + S_\alpha T$ are transformed into $Q^0{}_\alpha$ and $R^0{}_\alpha - S_\alpha(\dot{Q},x)/(1-\dot{Q}^2)^{1/2}$, respectively [see (8.45)]. Then the $\phi_{\rm Sch}$ is independent of t in agreement with the principle of quantum mechanics. Thus $\partial_t\phi_{\rm Sch} = 0$. Transforming it back to the Heisenberg representation, we obtain one of the four relations (10.1), namely, $[\partial_t - iH,\phi] = 0$.

In order to prove the rest of the relations (10.1), it is more convenient to use the expression (3.2) for ϕ. Then ϕ is a function of X_α, $R^0{}_\alpha + S_\alpha T$, and s. When we transform ϕ by $e^{ip^\mu x_\mu}$, X_α and $R^0{}_\alpha + S_\alpha T$ are reduced to $Q^0{}_\alpha + \dot{Q}_\alpha \times(-1+1(1-\dot{Q}^2)^{1/2})(\dot{Q}\cdot Q^0)/\dot{Q}^2$ and $R^0{}_\alpha$. The ϕ becomes independent of x_μ, in accordance with one of principles of the canonical formalism, namely,[20]

$$\partial_\nu e^{ip^\mu x_\mu}\phi e^{-ip^\mu x_\mu} = 0. \tag{10.3}$$

The relations (10.1) result from Eqs. (10.3).

XI. RELATIVISTIC CENTER OF MASS AND RELATIVISTIC DISPLACEMENT VECTOR

In the previous section VIII we have assigned the variable ξ appearing in the Hilbert space to be the center of the extended object. We will justify it in this section. At the same time we will make clear the definition of the three terms the center, the center of mass, and the relativistic displacement, often used in this paper.

What one calls the classical point mechanics describes the movement of the point to which the mass of the object is attached. There the inner structure of the object is completely ignored except the energy due to inner movements which is taken into account in the mass. In our theory of the extended object, the C-number part of the Heisenberg field ψ, namely, the order parameter $\phi(x)$, describes this inner structure. Here the center of the object is placed at the origin. Now we leave out this restriction and introduce variables ξ to denote the center of the object (later in this section these ξ will be identified with ξ introduced previously in Sec. VIII). Then, instead of $\phi(x)$, we will have a C-number solution of the Euler equation, which is a function of $x_\alpha - \xi_\alpha$. As a consequence, the Lagrangian integrated over space variables x is a function of $\dot{\xi}$ (note that $d\phi/dt = \dot{\xi}_\alpha\partial\phi/\partial\xi_\alpha$). Now the movement of ξ can be described in Lagrange's form of equations of motion

$$\frac{d}{dt}\frac{\partial}{\partial\dot{q}_i}L + \frac{\partial}{\partial q_i}L = 0.$$

In our case we can choose ξ and $\dot{\xi}$ for q and $\dot{q}$. Since our L is independent of ξ, the Lagrange equations are reduced to

$$\frac{d}{dt}\frac{\partial}{d\dot{\xi}}L(\dot{\xi}) = 0,$$

and therefore $\ddot{\xi} = 0$. The integrated Hamiltonian can be derived from the Lagrangian $L(\dot{\xi})$. Or, we may obtain it by first constructing the Hamiltonian density from the Lagrangian density and then integrating it [then we find that the Hamil-

tonian is also a function of $\dot{\xi}$ only, namely $H = H(\dot{\xi})$]. Then the movement of the center ξ can be described in Hamilton's form of equations of motion. Now we can quantize the variables ξ and $\dot{\xi}$. Namely, we transpose these Hamiltonian equations into matrix equations and at the same time construct the Schrödinger equation by introducing a Hilbert space. Then it is one of our purposes in this section to show that this Hilbert space is indeed identical to the one introduced previously in Sec. VIII.

We first deal with the spherical object. The C-number part of the Heisenberg field in the tree approximation, namely $\phi(x)$ [(2.6) in I], satisfies the Euler equation and is independent of time. Its center is placed at the origin. Then the static spherical object with center at $\xi^0 = (\xi^0{}_x, \xi^0{}_y, \xi^0{}_z)$ can be represented by the solution $\phi(x - \xi^0)$ of the Euler equation. When moving with the constant velocity $\dot{\xi}$, the object will be described by the function $\phi(x')$, where

$$x'_\alpha = x_\alpha - \xi^0{}_\alpha + \dot{\xi}_\alpha\left(-1 + \frac{1}{(1-\dot{\xi}^2)^{1/2}}\right) \times \frac{(\dot{\xi}\cdot x - \xi^0)}{\dot{\xi}^2} - \frac{\dot{\xi}_\alpha t}{(1-\dot{\xi}^2)^{1/2}}. \tag{11.1}$$

The x'_α given above are obtained from $x_\alpha - \xi^0{}_\alpha$ by a Lorentz transformation, and therefore $\phi(x')$ satisfies the Euler equation. In the classical mechanics of the point particle, the space coordinates are the variables depending on time and themselves satisfy the Lagrangian equation of motion. Then, in order to describe the uniform motion of the center of mass by the point mechanics, we rewrite x' [(11.1)] as

$$x' = x_\alpha - \xi_\alpha + \dot{\xi}_\alpha \times(-1 + 1/(1-\dot{\xi}^2)^{1/2})\,\dot{\xi}\cdot(x-\xi)/\dot{\xi}^2, \tag{11.2}$$

where ξ_α are $\xi^0{}_\alpha + \dot{\xi}_\alpha t$ and satisfy the Lagrangian equation of motion, namely $\ddot{\xi} = 0$. The x'_α [(11.2)] are identical to X' given in (2.10) of I if we replace Q_α in X'_α by ξ_α.

The energy and momenta of the classical object moving with velocity $\dot{\xi}$ are

$$H(\dot{\xi}) = M/(1-\dot{\xi}^2)^{1/2}, \quad P_\alpha(\dot{\xi}) = \dot{\xi}_\alpha M/(1-\dot{\xi}^2)^{1/2}. \tag{11.3}$$

They satisfy the relation $H(\dot{\xi})^2 = M^2 + P(\dot{\xi})^2$. The method for quantizing the variables ξ by using these four-momenta is familiar to everyone and can be seen, for example, in Ref. 14. We will not repeat it here. The Schrödinger functions, which we denote as $|\ \rangle$, have to satisfy following equations according to the general principles of quantum mechanics:

$$i\frac{\partial}{\partial \xi^\mu} = P_\mu |\ \rangle, \tag{11.4}$$

where P_μ are the matrix representation of $P_\mu(\dot{\xi})$ [(11.3)]. These equations are identical to Eq. (8.2) since P_μ used in Sec. VIII also satisfies the relation $H^2 = M^2 + P^2$. As a result the $|\ \rangle$ and ξ in this section are identical to the $|K\rangle$ and ξ introduced in Sec. VIII.

The P_α [(11.3)] can be obtained from P_α [(4.13)] by replacing $-Q_\alpha$ by $\dot{\xi}$. Consequently, the matrix representations constructed for P_α [(4.13)] in Sec. VIII, can be obtained by multiplying the matrices P_α obtained in this section by a factor -1 [in other words, the k_α in (8.2) have opposite signs to the k_α appearing in (11.4)]. The same is true for the case of the deformed object, as will be explained at the end of this section.

Before we start to deal with the deformed object, we need to explain the precise meaning of the center of mass and the relativistic displacement. We first consider them in classical mechanics.

The relativistic definition of the center of mass was given by Pryce and Møller in about 1948.[21] It is

$$X_\mu = \frac{P^\nu m_{\nu\mu}}{m^2} - \frac{P_\mu P^\nu m_{\nu 0}}{m^2 p^0} + \frac{P_\mu}{P^0}t. \tag{11.5}$$

Here we have used Pryce's notation as it is, except the sign of the metric tensor. The X_μ given above transform like four components of a Lorentz vector, as is mentioned by Pryce.[12] We then divide X_μ into two parts,

$$X_\mu = n_\mu + (P_\mu/P^0)(t - n_0) = n_\mu + v_\mu(t - n_0), \tag{11.6}$$

where

$$n_\mu = P^\nu m_{\nu\mu}/m^2 \tag{11.7}$$

and $v_\mu = P_\mu/H$. The n is evidently a Lorentz vector. The $v_\mu(t - n_0)$ in spite of its appearance is a vector. We can confirm this easily if we take into account the fact that the fourth component of X_μ is t. We will call n the relativistic displacement for the following reason. The n becomes independent of t and coincides with the center of the object, when the velocity vanishes, namely $P_\alpha/H = 0$ (or simply $P_\alpha = 0$). Then, the vector n indicates how much the center of the object is displaced from the origin. When the object is moving, the n signifies how far the trajectory of the center of mass deviates from the origin.

In case of our classical spherical object $\phi(x')$ with x' given in (11.2), the angular momenta $m_{\mu\nu} = \int(x_\mu T_{0\nu} - x_\nu T_{0\mu})\,d^3x$ consist of the orbital angular momenta only. As a result, the center of mass defined by (11.5) coincides with the center. Here the center means the set of mean values of the coordinates weighted by the density, instead of by the energy. When the object is not spherical, but does not rotate around its center, the center of mass still coincides with the center. However, this is no longer the case when the deformed object rotates.

Now we briefly deal with the deformed object. We start from the function describing the rotating deformed object whose center stays at the origin,

$$\phi(x,\varphi^0,\dot{\varphi}) = \exp(-(\varphi^0_\alpha + \dot{\varphi}_\alpha t)(x_\beta\partial_\gamma - x_\gamma\partial_\beta))\rho(x,y,z,|\dot{\varphi}|). \tag{11.8}$$

The φ^0_α determine the orientation of the deformed object, and $\dot{\varphi}_\alpha$ express three components of the angular velocity. Then we displace the center by d_α at $t = d_0$, and replace x_μ by $x_\mu - d_\mu = \tilde{x}_\mu$, namely,

$$\phi(\tilde{x},\varphi,\dot{\varphi}) = \exp(-(\varphi^0_\alpha - \dot{\varphi}_\alpha\tilde{t})(\tilde{x}_\beta\tilde{\partial}_\gamma - \tilde{x}_\gamma\tilde{\partial}_\beta))\tilde{\rho}(\tilde{x},\tilde{y},\tilde{z},|\dot{\varphi}|). \tag{11.9}$$

Then we accelerate the object and replace $\tilde{x}_\mu$ by the $\tilde{x}'_\mu$,

$$\tilde{x}'_\alpha = x_\alpha - d_\alpha + \dot{\xi}_\alpha\left(-1 + \frac{1}{(1-\dot{\xi})^{1/2}}\right)$$

$$\times\frac{\dot\xi\cdot(x-d)}{\dot\xi^2}-\frac{\dot\xi_\alpha(t-d_0)}{(1-\dot\xi^2)^{1/2}},$$

$$\tilde t'=\frac{(t-d_0)-\dot\xi\cdot(x-d)}{(1-\dot\xi^2)^{1/2}}. \tag{11.10}$$

When the velocity $\dot\xi$ is parallel to the spatial vector $d=(d_x, d_y, d_z)$, the trajectory of the center of the object passes through the origin. Let us introduce a Lorentz vector $\tau=(\tau_x,\tau_y,\tau_z,\tau_0)$, where

$$\tau_\alpha=\dot\xi_\alpha/(1-\dot\xi^2)^{1/2},\quad \tau_0=1/(1-\dot\xi^2)^{1/2}. \tag{11.11}$$

When the vector τ is parallel to the Lorentz vector $d=(d_x, d_y, d_z, d_0)$, the center of the object passes the origin at $t=0$. The $\tilde x'_\alpha$ become independent of d_μ, namely,

$$\tilde x'_\alpha=x_\alpha+\dot\xi_\alpha\left(-1+\frac{1}{(1-\dot\xi^2)^{1/2}}\right)\times\frac{\dot\xi\cdot x}{\dot\xi^2}-\frac{\dot\xi_\alpha t}{(1-\dot\xi^2)^{1/2}},$$

$$\tilde t'=\frac{t-\dot\xi\cdot x}{(1-\dot\xi^2)^{1/2}}+q, \tag{11.12}$$

where $q=d^\mu\tau_\mu$.

In the other extreme case, namely, when τ is perpendicular to d, the velocity $\dot\xi$ becomes perpendicular to the spatial vector (d_x, d_y, d_z) in the nonrelativistic limit. Then the Lorentz vector $d=(d_x, d_y, d_z, d_0)$ signifies, in a relativistic manner, how far the trajectory of the center of the object deviates from the origin. The $\tilde t'$ becomes independent of d_μ, and

$$\tilde x'_\alpha=x_\alpha-\eta_\alpha+\dot\xi_\alpha\left(-1+\frac{1}{(1-\dot\xi^2)^{1/2}}\right)\frac{\dot\xi\cdot(x-\eta)}{\dot\xi^2},$$

$$t'=\frac{t-\dot\xi x}{(1-\dot\xi^2)^{1/2}}, \tag{11.13}$$

where

$$\eta_\alpha=d_\alpha+\dot\xi_\alpha(t-d_0). \tag{11.14}$$

The d_μ are purely geometrical quantities. In spite of this we can express them in terms of four momenta P_μ and relativistic angular momenta $m_{\mu\nu}$ of the object $\phi(\tilde x',\varphi^0,\dot\varphi)$. We first construct P_μ and $m_{\mu\nu}$ of ϕ through the energy-momentum tensors $T_{\mu\nu}$. Then we find that P_μ and $m_{\mu\nu}$ are functions of $\dot\xi_\alpha,\dot\varphi_\alpha$, and d_μ. Then by reversing the functional relationship, we can express $\dot\xi_\alpha,\dot\varphi_\alpha$, and d_μ as functions of P_μ and $m_{\mu\nu}$. Finally, we find that

$$d_\mu=(P^\nu m_{\nu\mu}+q\hat M P_\mu)/\hat M^2, \tag{11.15}$$

where

$$\hat M=\dot\varphi^2 I+M(|\dot\varphi|).$$

These expressions are entirely parallel to the expression (6.3) for the quantum operators N_μ. When the vector d is perpendicular to τ, d_μ become

$$d_\mu=P^\nu m_{\nu\mu}/\hat M^2. \tag{11.16}$$

Therefore the d is the relativistic displacement vector n defined by (11.7). Correspondingly the η [(11.14)] is Pryce's center of mass (11.6). In the following we assume that τ is perpendicular to d, unless otherwise noted.

The angular momentum consists of the orbital and the intrinsic angular momenta, which we denote as $m^{\rm or}$ and $m^{\rm int}$. Correspondingly we can divide d_μ into two parts in a relativistic manner,

$$d_\mu=d^{\rm or}_\mu+d^{\rm int}_\mu, \tag{11.17}$$

where

$$d^{\rm or}_\mu=P^\nu m^{\rm or}_{\nu\mu}/\hat M^2 \tag{11.18}$$

and

$$d^{\rm int}_\mu=P^\nu m^{\rm int}_{\nu\mu}/\hat M^2. \tag{11.19}$$

In the same way we divide η_μ [(11.14)] into two parts,

$$\eta_\mu=\xi_\mu+\zeta_\mu, \tag{11.20}$$

where

$$\xi_\mu=\xi^0_\mu+\dot\xi_\mu t\quad\text{with}\quad \xi^0_\mu=d^{\rm or}_\mu-\dot\xi_\mu d^{\rm or}_0 \tag{11.21}$$

and

$$\zeta_\mu=d^{\rm int}_\mu-\dot\xi_\mu d^{\rm int}_0. \tag{11.22}$$

Here the η_0 an ξ_0 are defined as t. The ξ_μ [(11.21)] represent the center of the object (not the center of mass), since $m^{\rm or}$ in $d^{\rm or}$ [(11.18)] represents orbital angular momentum only.

We treat, as a matter of fact, $\dot\xi_\mu$ and $\dot\varphi_\alpha$ as two sets of independent variables. The η_μ and ζ_μ depend on both of $\dot\xi_\mu$ and $\dot\varphi_\alpha$. The ξ_μ, depend upon $\dot\xi_\mu$ only and are independent of $\dot\varphi_\alpha$. Therefore we will adopt ξ_μ as the variable conjugate to $\dot\xi$ when we later quantize $\dot\xi$. In other words the ξ_μ defined above in (11.21) will be identified as the ξ_μ introduced in Sec. VIII in order to construct the Hilbert subspace for P_μ.

Now we rapidly go through the quantization of the object $\phi(\tilde x',\varphi^0,\dot\varphi)$, where $\tilde x'$ are given in (11.13). The energy and momenta of the classical object ϕ moving with the velocity $\dot\xi$ and rotating with the angular velocity $\dot\varphi$ are

$$H(\dot\xi,\dot\varphi)=(\dot\varphi^2 I+M(|\dot\varphi|))/(1-\dot\xi^2)^{1/2},$$

$$P_\alpha(\dot\xi,\dot\varphi)=\dot\xi_\alpha(\dot\varphi^2 I+M(|\dot\varphi|))/(1-\dot\xi^2)^{1/2}. \tag{11.23}$$

Then the Schrödinger equation constructed out of this Hamiltonian $H(\dot\xi,\dot\varphi)$ becomes identical to Eq. (8.48). It means that S and $-Q$ are the matrix representations of $\dot\varphi$ and $\dot\xi$.[22] Now we choose the variables (11.21) as canonical conjugates of $P_\alpha(\dot\xi,\dot\varphi)$ [(11.23)], as we have just mentioned. Then we can easily identify $\hat N_\mu$, $\hat Q^0{}_\alpha$, N_μ, and $Q^0{}_\alpha$ as the matrix representations of $n^{\rm or}_\mu$, ξ^0_α, n_μ and $\eta^0_\alpha=\eta_\alpha-\dot\xi_\alpha t$. In conclusion, by quantizing the classical deformed object $\phi(\tilde x',\varphi^0,\dot\varphi)$, we obtain the vacuum value of the Heisenberg field ψ in the tree approximation. At the same time we confirm that the variables (11.21) are indeed identical to the ξ_μ introduced in Sec. VIII B. Further, the usage of the term "relativistic displacement operators" for N_μ, is now justified, since n_μ signify the displacement of the center of mass. Throughout the discussion given above, we did not mention the quantization of the variables φ^0 and $\dot\varphi$, since it is already done in Sec. VIII.

In the case when τ is not perpendicular to d, the matrix representation of d_μ becomes N_μ given by (6.3). Therefore, when we quantize the classical object moving along an arbitrary direction [namely $\phi(\tilde x',\varphi^0,\dot\varphi)$ with $\tilde x'$ (11.10)] we obtain the function (3.1) with N_μ given by (6.3), instead of (6.8). The center of mass operators Q_α are not altered. However, in the extreme case when τ is parallel to d, the ξ^0_α vanish as is seen

in (11.12). The quantization of such $\phi(\tilde{x}', \varphi^0, \dot{\varphi})$ is not possible. This is because such a solution ϕ does not contain sufficient variables to reproduce the Lorentz transformation.

The solution $\phi(\tilde{x}', \varphi^0, \dot{\varphi})$ with $\tilde{x}'$ given in (11.13), is the one which contains necessary and sufficient variables only in order to reproduce the Lorentz transformation.

XII. CONCLUSION

Within the last few years, a number of classical static solutions have been obtained for various Lorentz invariant nonlinear equations. By solving these equations as Euler equations in quantum field theory, we obtain solutions which one calls extended objects. Its order parameters are the classical static solutions mentioned previously. However, as is well known, the field theoretical solutions are not always stable because of the tunneling, even though the corresponding classical solution is static. Best examples are axially asymmetric solutions and have been discussed by many physicists in case of 1 + 1 dimensions, particularly by using ϕ^4 model. In quantum field theory, the unstable solutions cannot satisfy the canonical commutation relation. Consequently, in order that the solutions of the Euler equations satisfy the canonical commutation relation, it is not sufficient that the corresponding classical solutions are static.

To be precise, let us call only the solutions which satisfy the canonical commutation relation "Heisenberg field." Then it has been proven in this paper that there is a Lorentz covariant solution, namely a Heisenberg field, if the following conditions are satisfied. First, the order parameter $\phi(x)$ has to have a center and has to be either symmetrical or antisymmetrical for the simultaneous reflection of all space coordinates with respect to the center. Secondly, it has to be at an extremum.

In such a solution, namely Heisenberg field, there appear a certain number of classical quantum operators. In Sec. VIII we have given the Hilbert space for these quantum operators.

Each sector of the Hilbert space is a plane wave associated with an intrinsic angular momentum. It obeys the free equation, namely,

$$(\partial^\mu \partial_\mu + \hat{M})|l,m\rangle|K,l\rangle = 0,$$

where $\partial_\mu = \partial/\partial\xi^\mu$.

As a matter of fact, the sectors $|l,m\rangle|K,l\rangle$ are the representations of the Lorentz group. However, they differ from the representations most commonly used in particle physics. The latter can be obtained from ours, namely, from $|l,m\rangle|K,l\rangle$, by boosting $|l,m\rangle$. We can easily understand this fact by comparing ours with Weinberg's expressions for the wave functions with intrinsic spins $(\frac{1}{2}l,\frac{1}{2}l)$ (see Sec. VIII of Ref. 23).

Before we end this paper, we wish to mention a few examples of phenomena in which we should find the effects of the quantum operators. First we mention that nuclei are deformed extended objects. The author proposed the *j-j* coupling shell model in 1952 and subsequently has suggested the pairing trend of nucleons in nuclei in 1955.[24] In these two articles it has been shown that each nucleon is paired with another nucleon of the same kind in the ground state. One year later in 1956 such a pairing property was also found for electrons in superconducting materials. Nowadays these phenomena of pairing are explained in terms of symmetry breaking. In other words, in the ground state of nuclei and in superconductors, the vacuum value of $\psi\psi^+$ does not vanish. In the case of superconductors, it is a constant. In the case of nuclei, it is a local function of x, namely

$$\langle|\psi\psi^+|\rangle = v(x,y,z),$$

where $v = 0$ at $r > r_0$, and r_0 is the size of the nucleus. Therefore, in recent terminology, the nuclei are the extended objects. Now certain nuclei are deformed as is well known. Then they are the deformed extended objects.

The author wishes to add one further example; that is, an extremely deformed object called "string" in the particle physics. The Heisenberg fields for these two objects, namely nuclei and strings, are not real functions, and therefore their intrinsic angular momenta may be more complicated representations than the $\hat{S}$ used in this paper. Then we may observe the effect of somewhat more complicated fluctuation movement than what one can infer from this paper.

In the case of the string, the values of $x_\alpha\partial_\beta - x_\beta\partial_\alpha$ are far from vanishing. Since the total angular momentum $x_\alpha\partial_\beta - x_\beta\partial_\alpha + iM_{\alpha\beta}$ has to vanish for the static string, the contributions of $M_{\alpha\beta}$ are very important (by static, we mean that the order parameter is independent of time). Now, at each experimental observation, only one of matrix elements of the intrinsic angular momenta, ignoring the quantum linear momenta, will be observed according to the law of quantum mechanics, and not the average of all the matrix elements. Then the static string may be observed as if it is rotating (or spinning).

[1] T. D. Lee and G. C. Wick, Phys. Rev. D **9**, 2291 (1974). For other articles one may consult the following review articles: R. Rajaraman, Phys. Rep. **21**, 227 (1975); J. L. Gervais and A. Neveu, Phys. Rep. **23**, 237 (1976); R. Jackiw, Rev. Mod. Phys. **49**, 681 (1977); L. D. Faddeev and V. E. Korepin, Phys. Rep. **42**, 1 (1978).

[2] There are number of articles that appeared about 1975 or 1976. We may cite the following articles as two of the important ones: J. Goldstone and R. Jackiw, Phys. Rev. D **6**, 1486 (1975); J. L. Gervais and B. Sakita, Phys. Rev. D **11**, 2943 (1975).

[3] H. Matsumoto, N. J. Papastamatiou, H. Umezawa, and M. Umezawa, Phys. Rev. D **23**, 1339 (1981). If we use $T = (t - \dot{Q}\cdot x)/(1 - \dot{Q}^2)^{1/2}$, one obtains for χ a result which is clearly Lorentz covariant.

[4] H. Matsumoto, G. Semenott, M. Umezawa, and H. Umezawa, J. Math. Phys. **21**, 1761 (1980).

[5] M. Umezawa, Phys. Rev. D **24**, 1548 (1981). This article is referred to as I. Equations (3.2), (3.4), (10.3), (10.9), (10.10), and (10.13) of this article contain minor misprints. The correct ones are

$$P_\alpha = -\left\langle 0\middle|\int T_{0\alpha}\middle|0\right\rangle, \tag{3.2}$$

$$P_\alpha = \dot{Q}_\alpha H, \tag{3.4}$$

$$M(S) = \frac{1}{2}\int\sum_x (\partial_X\phi)^2 dX\,dY = \frac{1}{2}\int\sum_x (\partial_{X'}\rho)^2 dX'\,dY', \tag{10.3}$$

$$\delta\int\left\{-\frac{1}{2}S^2((x\partial_y - y\partial_x)\rho)^2 + \frac{1}{2}\sum\left(\frac{d}{dx}\rho\right)^2 + \frac{1}{2}m^2\rho^2 + V(\rho)\right\}dx\,dy = 0, \tag{10.9}$$

$$-\frac{1}{2}S^2\frac{d}{dS}\int((x\partial_y - y\partial_x)\rho)^2\,dx\,dy + \frac{d}{dS}\int\frac{1}{2}\sum\left(\frac{d}{dx}\rho\right)^2$$

$$+\frac{1}{2}m^2\rho^2 + V(\rho)dx\,dy = 0, \tag{10.10}$$

$$\frac{d}{dS}H = \frac{S}{(1-\dot{Q}^2)^{1/2}}\frac{d\hat{S}}{dS},$$

$$\frac{d}{dS}P = \dot{Q}\frac{S}{(1-\dot{Q}^2)^{1/2}}\frac{d\hat{S}}{dS}. \tag{10.13}$$

Also the ϕ appearing in between Eq. (10.10) and Eq. (10.13) should be replaced by ρ.

The note added in proof (b), is incorrect and in fact is not necessary.

[6]J. L. Jacquot and M. Umezawa, J. Math. Phys. **23**, 1693 (1982). This work was accomplished by Jacquot and Umezawa independently, who both obtained the same results. They decided to publish their results together. The manuscript above was written principally by Jacquot.

[7]In the case of the axial symmetric object, the I_z vanishes, if its axis lies on the z direction. Then $\hat{S}_z$ does not appear in $m_{\mu\nu}$. Such $m_{\mu\nu}$ do not form a Lorentz group by themselves. However, $m_{\mu\nu}$ do form effectively a Lorentz group when they are applied on ϕ. As regards Hilbert space, only a part of the Hilbert space introduced in Sec. VIII of this paper, namely the subspace with $\hat{S}_z = 0$, is sufficient [namely, $i(\mathring{\tau}\times\partial)_z$ of (8.27) is zero].

In order to give a relativistic appearance to $m_{\mu\nu}$, however, it might be better to formulate the axial symmetric object by choosing the orientation such that none of the I_α vanishes. Then we only have to take care that the relation $|\hat{S}| = Is$ has no significance when the direction of the vector $\hat{S}$ coincides with the axis of the object. All of $\hat{S}$, I, and S vanish suddenly on this line.

[8]In our case this constant vanishes, since we have assumed that $\phi(x)$ is symmetrical or antisymmetrical for the reflection with respect to the center of $\phi(x)$, namely, the origin of the coordinates.

[9]The $\hat{S}$ and W are related to each other by the relations

$$\hat{S}_\alpha = \exp\left(i\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\frac{V_0}{P_0}\right),$$

$$W_\alpha = \exp\left(-i\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\frac{V_0}{P_0}\right).$$

Consequently, the operators

$$\exp\left(-i\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\frac{V_0}{P_0}\right),$$

$$\hat{Q}_\alpha\exp\left(i\left(-1+\frac{1}{(1-\dot{Q}^2)^{1/2}}\right)\frac{(1-\dot{Q}^2)^{1/2}}{\dot{Q}^2}\frac{V_0}{P_0}\right)$$

commute with W_μ [see relation (6.11)].

[10]The coordinates X [(8.5) of I] are not influenced by the factor a since $N_\alpha - \dot{Q}_\alpha N_0$ are independent of a. However, the coordiante T [(8.6) of I] depends upon a and becomes $T - a\hat{M}$. Therefore the solution (3.1) is reduced to $\phi(x, y, z, t - a\hat{M}, R^0, S)$ when $\dot{Q}_\alpha$ vanishes.

[11]A. D. Fokker, *Relativitätstheorie* (Noordhoff, Groningen, Holland, 1929).

[12]M. H. L. Pryce, Proc. R. Soc. London, Ser. A **195**, 62 (1948); C. Møller, Communication of the Dublin Institute for Advanced Studies, Series A, No. 5 (1949).

[13]Two vectors $\hat{S}$ and S have a common direction, namely, the axis of the rotation. The $\hat{S}$ is the angular momentum vector and its length $|S|$ is independent of the orientation of the axis of rotation. In contrast with this, the S represents angular velocity and its length s is related to $|\hat{S}|$ by the relation $|\hat{S}| = sI$. The inertia I depends on the orientation of the axis of rotation. Therefore the length s also depends on the orientation of the axis of rotation. As a result, the length s in $\rho(x, y, z, s)$ indicates already along what direction ρ rotates.

[14]P. A. M. Dirac, *Quantum Mechanics*, 4th ed. (Oxford U. P., Oxford, England, 1958), p. 118.

[15]In the case of the extended object in 1 + 1 dimensions, this form of the Hilbert space was once used by J. Goldstone and R. Jackiw (see Ref. 2).

[16]The expression (4.3) of I can be rewritten as $iH\partial/\partial P_\mu$ and is of the Heisenberg representation. See T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[17]We will show that the vectors R and $\hat{S}$ have the same direction at each matrix element, unless one of them vanishes. To do this it is sufficient if we prove this for the case when $l = l'$. Let us transform the representation $|l,m\rangle_{\rm Sch}$ into $|l,m\rangle^*_{\rm Sch}$ in which $i(\mathring{\tau}\times\partial)_\varphi$ [(8.28)] is diagonal; that is to say,

$$i(\mathring{\tau}\times\partial)_\varphi|l,m\rangle^*_{\rm Sch} = m|l,m\rangle^*_{\rm Sch}.$$

Then, from the canonical relation (8.29) we can easily prove that the matrix elements $\int^*\langle lm'|\,|\varphi\,|\,|l,m\rangle^*\,d\Omega$ vanish unless $m = m'$. The other two components of $\hat{S}$, namely, the ones orthogonal to the direction of φ vanish when $m = m'$. Thus only the component $(\mathring{\tau}\times\partial)_\varphi$ is nonzero at $m = m'$, and therefore $\hat{S}$ is parallel to R^0.

[18]The set $\hat{N} = (N_0, N_x, N_y, N_z)$ is a relativistic vector. It expresses how far the trajectory of the center of the object (the center of the density) deviates from the origin. It is the relativistic displacement operator of the center of the object, while the vector N [(6.8)] is the relativistic displacement operator of the center of mass. $\hat{N}$ differs from N because the energy density becomes asymmetric with respect to the trajectory of the center on account of the interference between the uniform translational motion and the intrinsic rotational motion.

[19]The s is related to $|\hat{S}|$ through the relation $|\hat{S}| = SI$, where I is a function of s. Then s is a function of $|\hat{S}|$ and therefore commutes with $|\hat{S}|$. The fact that I, namely the moment of inertia, is a function of $s = (\Sigma_\alpha S^2{}_\alpha)^{1/2}$ and not a function of the S_α themselves, can be explained as follows. When we rotate the object (8.18) by one of the $\hat{S}_\alpha$, the intrinsic angular momenta of the object, namely the $\hat{S}_\alpha$ themselves, are also rotated. The S and $\hat{S}$ have the same direction and therefore S is also rotated. Finally, since the object and S are both rotated, the moment of inertia along the direction of S is not altered. That is to say, I commutes with $\hat{S}_\alpha$. Thus I is a function of s only.

[20]W. Heisenberg and W. Pauli, Z. Phys. **47**, 151 (1928). In particular see relation (25).

[21]Pryce gave expressions (11.5) for the composite system of free particles. I believe, however, that these expressions are accepted for any system nowadays.

[22]The Hamiltonian (11.21) determines the total sectors $|K,l,m\rangle$. However, there remains infinitely many ways of writing each $|K,l,m\rangle$ as a product of two subsectors. In Sec. VIII, we have adopted $|K,l\rangle$ [(8.13)] and $|l,m\rangle_{\rm Sch}$ [(8.33)] as these two subsectors by choosing $|l,m\rangle$ to be independent of time. Then, for example, we may transform these $|K,l\rangle$ and $|l,m\rangle_{\rm Sch}$ into another set $|K,l,w\rangle$ and $|l,m,w\rangle$, where $|K,l,w\rangle = e^{iwT}|K,l\rangle$ and $|l,m,w\rangle = e^{-iwt}|l,m\rangle$. Here w is an arbitrary function of K and l. The total sectors $|K,l,m\rangle$ are not altered by this transformation. The operators P_μ, Q_α, and S_α are still the matrix representations of $i\partial/\partial\xi^\mu$, $\dot{\xi}_\alpha$, and $\dot{\varphi}_\alpha$. If one chooses the following particular value for w,

$$w = (M(0) - \{l(l+1) + M(l)\})/(1-\dot{Q}^2)^{1/2},$$

the $|K,l,w\rangle$ obey the equation which is independent of l, that is,

$$-i\partial_t|K,l,w\rangle = (M(0)/(1-\dot{Q}^2)^{1/2})|K,l,w\rangle.$$

Then $|K,l,w\rangle$ are independent of l, and are therefore equal to $|K,0,w\rangle$.

[23]Steven Weinberg, Phys. Rev. **133**, B1318 (1964).

[24]Minoru Umezawa, Prog. Theor. Phys. **8**, 509 (1952) and Proc. R. Soc. London, Ser. A **232**, 88 (1955).

An SL(2,C)-invariant representation of the Dirac equation. II. Coulomb Green's function

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The Kepler problem for the Klein–Gordon type wave equation

$$\{\Pi_\mu\Pi_\mu + m^2 + ie\boldsymbol{\sigma}\cdot(\mathscr{E} + i\mathbf{B})\}\phi = 0,$$

investigated earlier [J. Math. Phys. **23**, 1179 (1982)] and proven to be equivalent to the conventional Dirac equation, is discussed. In this equation ϕ is a 2×1 Pauli spinor and σ_a, $a = 1, 2, 3$, are the usual 2×2 Pauli spin matrices. Quite simple expressions for the bound state Coulomb wavefunctions and for the Coulomb Green's function are obtained by exploiting the concept of "coupling constant eigenfunction." To facilitate the direct use of these simple expressions in Coulomb calculations, a stationary state perturbation theory appropriate for the Klein–Gordon type wave equation itself is described.

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I. INTRODUCTION

We will here investigate the quantum mechanical Kepler problem for the Klein–Gordon type equation

$$[\Pi_\mu\Pi_\mu + m^2 + ie\boldsymbol{\sigma}\cdot(\mathscr{E} + i\mathbf{B})]\phi = 0,$$
$$\Pi_\mu = -i\partial_\mu - eA_\mu, \tag{1.1}$$

considered previously.[1] In this equation ϕ is a 2×1 Pauli spinor, and σ_a, $a = 1,2,3$, are the usual 2×2 Pauli spin matrices. It has been shown in Ref. 1 that Eq. (1.1), equipped with the inner product

$$(\phi_b;\phi_a) \equiv i\int d^3r\, \bar{\phi}_b \overleftrightarrow{\Pi}_4 \phi_a, \tag{1.2}$$

provides a description of a charged spin-$\frac{1}{2}$ particle which is entirely equivalent to the conventional Dirac equation. The "dual" wavefunction appearing in the inner product (1.2) is analogous to the Dirac $\bar{\psi} = \psi^\dagger\gamma_4$ and is defined as[2]

$$\bar{\phi} \equiv \phi^\dagger \tau_\mu \overleftarrow{\Pi}_\mu/2m^2,$$
$$\tau_a \equiv \sigma_a, \quad a = 1,2,3, \quad \tau_4 \equiv i. \tag{1.3}$$

It has been shown in Ref. 1 that the structure $\bar{\phi}_b\phi_a$ is invariant under a proper Lorentz transformation.

Interest in Eq. (1.1) stems from the fact, proven in Ref. 1, that Feynman rules for Eq. (1.1) are just the rules of scalar electrodynamics aside from the replacement of the factor $-ie(p_{f\mu} + p_{i\mu})$ for the one-photon vertex of ordinary scalar electrodynamics by the new factor

$$-ie[p_{fv}(1 + i\sigma)_{v\mu} + (1 + i\sigma)_{\mu v}p_{iv}]$$

for the one-photon vertex associated with Eq. (1.1). Here $(1)_{\mu\nu} \equiv \delta_{\mu\nu}$ is the metric tensor in Minkowski space, and $\sigma_{\mu\nu}$ is the self-dual spin tensor

$$\sigma_{\mu\nu} \equiv \left(\begin{array}{ccc|c} 0 & \sigma_3 & -\sigma_2 & \sigma_1 \\ -\sigma_3 & 0 & \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 & 0 & \sigma_3 \\ \hline -\sigma_1 & -\sigma_2 & -\sigma_3 & 0 \end{array}\right) \tag{1.4}$$

made up of the Pauli matrices. Scalar and spinor electrodynamics can thus be treated in a unified fashion.[3] Because of the small dimension of the matrices involved, this new "scalar formalism" for spin-$\frac{1}{2}$ particles may prove to be quite convenient for applications. A program of calculations of radiative corrections in quantum electrodynamics is planned in which the new scalar formalism will be used. The material on the quantum mechanical Kepler problem reported here is a result of a preliminary step in this program of calculations.

The Coulomb problem of the conventional Dirac equation is already familiar from many earlier treatments.[4] Accordingly, our report can be brief as regards most of the familiar algebra of Laguerre polynomials, etc. The new Kepler problem contained in Eq. (1.1) has much to recommend it. The expressions for the Coulomb wavefunctions and for the Coulomb Green's function of the new wave equation have relatively simple, compact structures. It is expected that these relatively simple expressions will facilitate efficient calculations involving the relativistic Coulomb problem.

In Sec. II the concept of "coupling constant eigenfunction" is considered. Through the use of this concept it is possible to write down an eigenfunction expansion, Eq. (2.19), of the relativistic Coulomb Green's function in whch only a discrete sum of states appears. Such a discrete expansion for the nonrelativistic Coulomb Green's function has been known for some time. The momentum space form of such an expansion was obtained by Schwinger,[5] who exploited the O(4) invariance of the nonrelativistic Kepler problem. The coordinate space analog of Schwinger's nonrelativistic result was discussed by Hostler, who used the coupling constant eigenfunction concept.[6] That this coupling constant eigenfunction concept could yield a discrete expansion also in the relativistic Kepler problem does not seem to have been exploited previously. The new expansion, Eq.(2.19), is expected to be quite useful for applications.

Conventional "frequency eigenfunctions" are investigated next (Sec. III). These are found to form an overcom-

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plete set of states when regarded as a basis for expansion of functions, such as the Coulomb Green's function, which are independent of time. This phenomenon is characteristic of Klein–Gordon type wave equations and has been encountered before.[7] After a digression investigating the analogues of orthogonality and completeness relations for the overcomplete set of states, we arrive at the expected form, Eq. (3.9), of the stationary state eigenfunction expansion of the relativistic Coulomb Green's function.

A brief account of stationary state perturbation theory for Eq. (1.1) is included for completeness sake (Sec. IV). This perturbation theory is quite different from the usual Rayleigh–Schrödinger perturbation theory. The method used is a form of the contour integration approach of Kato,[8] as extended earlier for application to the Bethe–Salpeter equation.[9] The concept of "reduced Green's function," introduced earlier in a different context,[10] is found to have a useful organizing effect here as well. The relevant part of this material needed to do stationary state perturbation theory is summarized in a notation tailored to the specific equation of interest, our Eq. (1.1).

II. COUPLING CONSTANT EIGENFUNCTION EXPANSION

When Eq. (1.1) is written out in full for the special case of an external Coulomb potential, $V = -\mathbb{Z}\alpha/r$, we find:

$$O\phi = 0,$$

$$O \equiv -\nabla^2 + \eta^2 - \frac{2E\alpha}{r} - \frac{\alpha^2 + i\alpha\boldsymbol{\sigma}\cdot\mathbf{r}}{r^2}, \tag{2.1}$$

$$\eta \equiv (m^2 - E^2)^{1/2}.$$

For the physical situation of interest we have the relations $\alpha = \alpha = \mathbb{Z}\alpha$. However, in this section we shall treat a somewhat generalized Kepler problem in which α and α are two independent coupling constants.

Using techniques familiar from the solution of the conventional Dirac equation,[4] we can rewrite O in the form

$$O = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \eta^2 - \frac{2E\alpha}{r} + \frac{K^2 - K - \alpha^2 - i\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}}{r^2}, \tag{2.2}$$

$$K \equiv \boldsymbol{\sigma}\cdot\mathbf{L} + 1. \tag{2.3}$$

The operator K is self-adjoint and obeys the anticommutation relation

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}K + K\boldsymbol{\sigma}\cdot\hat{\mathbf{r}} = 0. \tag{2.4}$$

The eigenfunctions of K, $Y_{IJM}(\hat{\mathbf{r}})$ are the simultaneous eigenfunctions of $\mathbf{s}\cdot\mathbf{s}$, $\mathbf{L}\cdot\mathbf{L}$, $\mathbf{J}\cdot\mathbf{J}$, and J_z.[11] Next we introduce an operator S analogous to an operator of Biedenharn[12] and Martin and Glauber[13] introduced in connection with the conventional Dirac equation. The definition is

$$S \equiv \cosh(\tfrac{1}{2}\theta) + i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\sinh(\tfrac{1}{2}\theta), \tag{2.5}$$

$$\theta \equiv \tanh^{-1}(\alpha/K). \tag{2.6}$$

Note that θ is self-adjoint, and that θ is diagonal in the same representation, with eigenfunctions $Y_{IJM}(\hat{\mathbf{r}})$, as K. Also, by virtue of Eq. (2.4), $\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$ anticommutes with any odd function of θ and commutes with any even function of θ. By means of a similarity transformation employing the operator S, O can be written in terms of a self-adjoint operator which is again diagonal in the same representation as K:

$$O = S\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \eta^2 - \frac{2E\alpha}{r} - \frac{1 - (2\gamma+1)^2}{4r^2}\right] S^{-1}, \tag{2.7}$$

$$\gamma + \tfrac{1}{2} \equiv (K^2 - \alpha^2)^{1/2} - \tfrac{1}{2}\epsilon(K), \tag{2.8}$$

$$\epsilon(K) = +1, \quad K > 0, \quad \epsilon(K) = -1, \quad K < 0.$$

The self-adjoint operator γ is diagonal in the representation Y_{IJM} and has eigenvalues $\gamma = [(l+1)^2 - \alpha^2]^{1/2} - 1$ for $l = J - \frac{1}{2}$ and $\gamma = [l^2 - \alpha^2)^{1/2}$ for $l = J + \frac{1}{2}$. Equation (2.1) now takes the form

$$\left(-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \eta^2 - \frac{2E\alpha}{r} - \frac{1 - (2\gamma+1)^2}{4r^2}\right) S^{-1}\phi = 0. \tag{2.9}$$

It is at this point that the concept of "coupling constant eigenfunction" comes in. Equation (2.9) can be rearranged to make it look like an eigenvalue equation for α:

$$\Lambda\phi_\Lambda = \alpha\phi_\Lambda, \tag{2.10}$$

$$\phi_\Lambda = r^{-1/2}S^{-1}\phi, \tag{2.11}$$

and

$$\Lambda = \left(\frac{r}{2E}\right)^{1/2} \times\left(-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \eta^2 - \frac{1 - (2\gamma+1)^2}{4r^2}\right)\left(\frac{r}{2E}\right)^{1/2}. \tag{2.12}$$

For a preassigned real energy E the operator Λ will be a self-adjoint operator, diagonal in the representation $Y_{IJM}(\hat{\mathbf{r}})$, whose eigenfunctions form a complete orthonormal set. When E is restricted to the range $0 < E < m$, this set is a discrete set. We thus obtain a discrete orthonormal basis of Coulomb functions. Aside from the operator factor $r^{-1/2}S^{-1}$ of Eq. (2.11), these Coulomb functions have the physical intepretation of the stationary state wavefunctions, which for suitable values of α (the eigenvalues of Λ) would belong to the given energy E. To solve the eigenvalue problem (2.10) is a standard exercise in quantum mechanics. The resulting eigenfunctions and eigenvalues are[14]

$$\phi_{\Lambda nlJM} = (2\eta)^{3/2}[(n-1)!/(n+2\gamma)!]^{1/2}(2\eta r)^{\gamma - 1/2} \times e^{-\eta r} L_{n-1}^{2\gamma+1}(2\eta r) Y_{IJM}(\hat{\mathbf{r}}), \tag{2.13}$$

$$\alpha = (\gamma + n)\eta/E, \quad n = 1,2,3,\cdots, \quad 0 < E < m. \tag{2.14}$$

The normalization is

$$\int d^3r\, \phi^\dagger_{AB}\phi_{AA} = \delta_{B,A}, \tag{2.15}$$

where B and A each refer to a particular set of values for the quantum numbers $nlJM$. In Eqs. (2.13) and (2.14) we have used the same symbol γ to denote the eigenvalue as we had earlier [Eq. (2.8)] used to denote the operator. Note that our quantum number n runs over the range $n = 1,2,3,\cdots$ independently of the quantum number l.

The utility of the discrete basis (2.13) is illustrated by the following application to the Coulomb Green's function. The Green's function associated with Eq. (2.1) may be defined through the abstract operator statement

$$\mathcal{G} \equiv -1/O \tag{2.16}$$

where we assume for the moment that $0 < E < m$. Referring to Eqs. (2.7) and (2.12), we can write

$$\mathcal{G} = S\left(\frac{r}{2E}\right)^{1/2} \frac{-1}{\Lambda - \alpha}\left(\frac{r}{2E}\right)^{1/2} S^{-1}. \tag{2.17}$$

A discrete expansion is obtained by inserting a complete set of eigenstates of Λ in Eq. (2.17):

$$\mathcal{G} = \sum_{A} S\left(\frac{r}{2E}\right)^{1/2} \frac{-|\phi_{\Lambda A}\rangle\langle\phi_{\Lambda A}|}{(\gamma+n)\eta/E - \alpha}\left(\frac{r}{2E}\right)^{1/2} S^{-1}. \tag{2.18}$$

The coordinate space Coulomb Green's function

$$\mathcal{G}(\mathbf{r}_2,\mathbf{r}_1,E) \equiv \langle \mathbf{r}_2|\mathcal{G}|\mathbf{r}_1\rangle$$

can be read off from Eq. (2.18):

$$\mathcal{G}(\mathbf{r}_2,\mathbf{r}_1,E) = -\sum_{nlJM}\left(\frac{r_2}{2E}\right)^{1/2} \frac{T_2\phi_{\Lambda nlJM}(\mathbf{r}_2)\phi^{\dagger}{}_{\Lambda nlJM}(\mathbf{r}_1)T_1}{(\gamma+n)\eta/E-\alpha}\left(\frac{r_1}{2E}\right)^{1/2}, \tag{2.19}$$

$$T_{2,1} \equiv \cosh(\tfrac{1}{2}\theta) + i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}_{2,1}\sinh(\tfrac{1}{2}\theta). \tag{2.20}$$

We adopt the convention that θ shall signify the operator when it appears in S, but the eigenvalue with it appears in T.[15] Although Eq. (2.19) has been proven only for E in the restricted range $0 < E < m$; this restriction can be lifted by use of the theory of analytic continuation.

Equation (2.19) is the discrete expansion of the relativistic Coulomb Green's function referred to in the Introduction. Its compactness and relative simplicity should make possible relatively efficient calculations in applications involving the relativistic Coulomb Green's function. By substituting Eq. (2.19) in Eq. (2.41) of Ref. 1, a corresponding result for the usual Dirac Coulomb Green's function will be obtained. However, it is expected that, in order to fully exploit the simple form of the new representation, one should work entirely within the framework of the Klein–Gordon type wave equation (1.1) itself.

For completeness we record here the partial wave expansion

$$\begin{aligned}\mathcal{G}(\mathbf{r}_2,\mathbf{r}_1,E) = {} & \sum_{lJM} T_2 Y_{lJM}(\hat{r}_2)Y^{\dagger}_{lJM}(\hat{r}_1)T_1 \\ & \times(-2\eta)(\rho_>\rho_<)^{-1}\Gamma(\gamma+1-E\alpha/\eta) \\ & \times W_{E\alpha/\eta,\gamma+1/2}(\rho_>)\mathcal{M}_{E\alpha/\eta,\gamma+1/2}(\rho_<),\end{aligned} \tag{2.21}$$

$$\rho_> = 2\eta r_>, \quad \rho_< = 2\eta r_<, \quad r_> = \max(r_2,r_1), \quad r_< = \min(r_2,r_1),$$

of the relativistic Coulomb Green's function. The factors $T_{2,1}$ here are as defined by Eq. (2.20). The functions W and $\mathcal{M}$ are the irregular and regular solutions, respectively, of Whittaker's equation.[16] The derivation of Eq. (2.21) involves solving the partial wave radial Green's function equation in a standard way.

III. FREQUENCY EIGENFUNCTION EXPANSION

A. General formalism

In order to do physics, we must eventually make contact with the usual stationary state wavefunctions, referred to here as "frequency eigenfunctions." We therefore return to the customary point of view in which the strength of the coupling to the external potential is given, and the energy E is a parameter ("eigenvalue") to be determined. For purposes of this subsection we can just as well treat an arbitrary static field.

We shall write the frequency eigenfunctions obeying Eq. (1.1) in the form $\psi_A = \phi_A \exp(-i\epsilon_A E_A t)$ explicitly exhibiting the time dependence of the state ψ_A. The subscript A represents a collection of quantum numbers, such as the set $nlJM$ of Sec. II. We assume $E_A > 0$. The sign of the energy is indicated explicitly through the factor $\epsilon_A \equiv \pm 1$. The stationary state wavefunctions are assumed to be normalized according to

$$(\psi_B;\psi_A) = \delta_{B,A}. \tag{3.1}$$

The inner product is here the inner product of Eq. (1.2). Because of current conservation, the left-hand side of Eq. (3.1) is independent of the time at which the inner product is formed.

We would like to use the stationary state wavefunctions to expand the Green's function $\mathcal{G}(\mathbf{r}_2,\mathbf{r}_1,E)$. We here encounter a problem characteristic of Klein–Gordon type equations.[7] A completeness relation for the time-independent functions ϕ_A must be sought in the expansion theorem

$$f(\mathbf{r},t) = \sum_{A} c_A \psi_A, \tag{3.2}$$

expressing an arbitrary time-dependent solution of the wave equation as a linear superposition of the full time-dependent frequency eigenfunctions ψ_A. To obtain the completeness relation for the functions ϕ_A, we start by expressing the expansion coefficients in Eq. (3.2) in terms of f. By use of the orthogonality relation (3.1), we find $c_A = i\int d^3r\, \bar{\psi}_A \overleftrightarrow{\Pi}_4 f$. Now we substitute this expression for c_A back in Eq. (3.2) and set $t = 0$ throughout. The result is the identity

$$\begin{aligned} f(\mathbf{r}_2,0) = {} & \int d^3r_1 \sum_A \phi_A(2)\bar{\phi}_A(1)[-\epsilon_A E_A + 2eV(1)]f(\mathbf{r}_1,0) \\ & - i\int d^3r_1 \sum_A \phi_A(2)\bar{\phi}_A(1)\dot{f}(\mathbf{r}_1,0). \end{aligned} \tag{3.3}$$

In this equation the time-independent object $\bar{\phi}_A$ is interpreted in conformity with Eq. (1.3):

$$\bar{\phi}_A \equiv \phi^{\dagger}_A[-\epsilon_A E_A + eV + \boldsymbol{\sigma}\cdot(-i\overleftarrow{\nabla} - e\mathbf{A})]/2m^2.$$

For a general solution of a second order wave equation $f(\mathbf{r},0)$ and $\dot{f}(\mathbf{r},0)$ can be specified quite independently. In order for the identity (3.3) to remain true when $f(\mathbf{r}_1,0)$ and $\dot{f}(\mathbf{r}_1,0)$ are independent functions, we must have the relation

$$\sum_A \phi_A(2)\bar{\phi}_A(1) = 0. \tag{3.4}$$

This relation is designed to make the $f(\mathbf{r}_1,0)$ terms drop out of Eq. (3.3), as is required by the fact that we may set $f(\mathbf{r}_1,0)=0$, while $f(\mathbf{r}_1,0)$ remains arbitrary. Returning to the case of a general f, we find that the identity (3.4) eliminates the potential terms as well as the $f(\mathbf{r}_1,0)$ terms from Eq. (3.3). The resulting form of Eq. (3.3), which now involves only the arbitrary function $f(\mathbf{r}_1,0)$, is equivalent to the completeness relation

$$\delta^3(2,1) = -\sum_A \phi_A(2)\bar{\phi}_A(1)\epsilon_A E_A. \tag{3.5}$$

Equations (3.4) and (3.5) express the fact, mentioned in the Introduction, that the stationary state wavefunctions ϕ_A form an overcomplete set. Equation (3.4) states that the ϕ_A are linearly dependent, while Eq. (3.5) shows that they, nevertheless, form a complete set.

We have now assembled all the tools needed to derive the frequency eigenfunction expansion of the Green's function. The Green's function equation to be solved is

$$[-(E-eV_2)^2+\vec{\Pi}_2\cdot\vec{\Pi}_2+m^2+ie\boldsymbol{\sigma}\cdot(\mathscr{E}_2+i\mathbf{B}_2)]\mathscr{G}(2,1) = -\delta^3(2,1). \tag{3.6}$$

We permit E to have any complex value not in the frequency eigenvalue spectrum of the system. This Green's function equation is used in conjunction with the equation

$$\bar{\phi}_A(2)[-(\epsilon_A E_A - eV_2)^2+\overleftarrow{\Pi}_2\cdot\overleftarrow{\Pi}_2+m^2 + ie\boldsymbol{\sigma}\cdot(\mathscr{E}_2+i\mathbf{B}_2)] = 0 \tag{3.7}$$

of the dual state $\bar{\phi}_A$ to derive the identity

$$\int d^3r_2\,\bar{\phi}_A(2) \times(E-\epsilon_A E_A)(-E-\epsilon_A E_A+2eV_2)\mathscr{G}(2,1) = -\bar{\phi}_A(1). \tag{3.8}$$

The desired eigenfunction expansion is obtained from Eq. (3.8) by first dividing through by $(E-\epsilon_A E_A)$, then multiplying through by $\phi_A(3)$, and finally, performing a sum over states:

$$\mathscr{G}(2,1) = -\sum_A \phi_A(2)\bar{\phi}_A(1)/(E-\epsilon_A E_A). \tag{3.9}$$

Note that *both* identities (3.4) and (3.5) are needed to obtain this result. In spite of this unusual mathematics, our final result, Eq. (3.9), has a familiar form. The frequency eigenfunction expansion (3.9) will play a role in the stationary state perturbation theory to be presented in Sec. IV.

B. Simultaneous parity and frequency eigenfunctions

A parity operator $\mathscr{P}$ can be defined for our system through the equations

$$\mathscr{P}\phi = -2m\bar{\phi}^{\dagger}(-\mathbf{r},t) \tag{3.10}$$

and

$$\bar{\phi}\overleftarrow{\mathscr{P}} = (-1/2m)\phi^{\dagger}(-\mathbf{r},t). \tag{3.11}$$

The spinors $\mathscr{P}\phi$ and $\bar{\phi}\overleftarrow{\mathscr{P}}$ describe a new state of the system obtained by reflecting the given state through the origin. Accordingly, $\mathscr{P}\phi$ again obeys the wave equation (1.1), but this time with the reflected potentials: $\mathscr{P}V = V(-\mathbf{r},t)$ and $\mathscr{P}\mathbf{A} = -\mathbf{A}(-\mathbf{r},t)$. Equations (3.10) and (3.11) may be derived by a study of the reflection properties of Eq. (1.1). The phase in Eq. (3.10) has been chosen for compatibility with the corresponding formula $\mathscr{P}\psi = \gamma_4\psi(-\mathbf{r},t)$ of the conventional Dirac equation, to which our formalism is equivalent.

It has already been recognized that, aside from the operator factor $r^{1/2}S$ of Eq. (2.11), the states ϕ_{AA} of Eq. (2.13) double as frequency eigenfunctions of our system. Except for the levels with $n=1$, $l=J-\frac{1}{2}$, however, the states in question, $r^{1/2}S\phi_{AA}$, are not parity eigenfunctions. In general, in order to obtain simultaneous parity and frequency eigenfunctions linear combinations of the degenerate levels $r^{1/2}S\phi_{Anl=J-(1/2)JM}$ and $r^{1/2}S\phi_{An-1\,l=J+(1/2)JM}$ must be sought which diagonalize $\mathscr{P}$. These calculations, which are rather long and tedious, are described in Appendix A. The final expressions for the simultaneous parity and frequency eigenfunctions turn out to be quite simple. These are

$$\chi = \frac{\eta}{(2m\alpha)^{1/2}} r^{1/2}S\left[\left(\frac{E|K|}{\nu m}+1\right)^{1/2}\phi_{An\,J-1/2} + i\left(\frac{E|K|}{\nu m}-1\right)^{1/2}\phi_{A\,n-1\,J+1/2}\right], \tag{3.12}$$

$$\bar{\chi} = -\frac{1}{2m}\frac{\eta}{(2m\alpha)^{1/2}}\left[\left(\frac{E|K|}{\nu m}+1\right)^{1/2}\phi^{\dagger}_{An\,J-1/2} + i\left(\frac{E|K|}{\nu m}-1\right)^{1/2}\phi^{\dagger}_{A\,n-1\,J+1/2}\right]S^{-1}r^{1/2}, \tag{3.13}$$

$$\mathscr{P} = (-1)^{J-1/2}, \quad n = 1,2,3,4,\cdots,$$

and

$$\chi = \frac{\eta}{(2m\alpha)^{1/2}} r^{1/2}S\left[\left(\frac{E|K|}{\nu m}+1\right)^{1/2}\phi_{A\,n-1\,J+1/2} - i\left(\frac{E|K|}{\nu m}-1\right)^{1/2}\phi_{An\,J-1/2}\right], \tag{3.14}$$

$$\bar{\chi} = -\frac{1}{2m}\frac{\eta}{(2m\alpha)^{1/2}}\left[\left(\frac{E|K|}{\nu m}+1\right)^{1/2}\phi^{\dagger}_{A\,n-1\,J+1/2} - i\left(\frac{E|K|}{\nu m}-1\right)^{1/2}\phi^{\dagger}_{An\,J-1/2}\right]S^{-1}r^{1/2}, \tag{3.15}$$

$$\mathscr{P} = (-1)^{J+1/2}, \quad n = 2,3,4,\cdots.$$

In these equations we have reverted back to the Coulomb problem with equal coupling constants $\alpha = \acute{\alpha} = \mathbb{Z}\alpha$. Other notations are $\nu = (K^2-\alpha^2)^{1/2}$, $\eta = (m^2-E^2)^{1/2}$, $E = m(1+\alpha^2/(\gamma+n)^2)^{-1/2}$. Only the radial quantum number, n or $n-1$, and the orbital angular momentum quantum number, $l = J\pm\frac{1}{2}$, are indicated explicitly. The states (3.12) through (3.15) have been normalized with respect to the metric (1.2):

$$i\int d^3r\bar{\chi}_B\,\overleftrightarrow{\Pi}_4\chi_A = \delta_{A,B}. \tag{3.16}$$

It is assumed that the functions ϕ_{AA} are precisely the functions (2.13) with normalization (2.15). Although Eqs. (3.12) and (3.13) were derived for $n=2,3,4,\cdots$ (see Appendix A); it turns out that the $n=1$, $l=J-\frac{1}{2}$ levels are described correctly also: For the $n=1$, $l=J-\frac{1}{2}$ levels the second terms of Eqs. (3.12) and (3.13) are interpreted as zero.

IV. STATIONARY STATE PERTURBATION THEORY FOR A KLEIN–GORDON TYPE EQUATION

To discuss the perturbation theory of Eq. (2.1) we write it in the form ($a = \mathcal{b} = \mathbb{Z}\alpha$ again):

$$[T(E) + V(E)]\chi_0 = 0, \tag{4.1}$$

where

$$T(E) = -\nabla^2 + m^2 - E^2 \tag{4.2}$$

and

$$V(E) = -2Ea/r - (a^2 + ia\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})/r^2. \tag{4.3}$$

A perturbation theory different from the usual Rayleigh–Schrödinger perturbation theory is required because of the complicated energy dependence of the terms in Eq. (4.1). Equation (4.1) has a form exactly like that of a Bethe–Salpeter equation. The contour integration techniques of Kato[8] have been developed into an efficient perturbation formalism for the Bethe–Salpeter equation,[9] and this formalism is applicable in its entirety to the present problem. This formalism exploits the pole structure of the frequency eigenfunction expansion (3.9), which we rewrite here as

$$\mathcal{g}_0(E) = -\sum_B \mathcal{P}_{0B}/(E - E_{0B}), \tag{4.4}$$

$$\mathcal{P}_{0B} = |\chi_{0B}\rangle\langle\overline{\chi_{0b}}|, \quad \overline{\langle\chi_{0B}|} \equiv \langle\chi_{0B}|\tau_\mu \Pi_\mu/2m^2. \tag{4.5}$$

The operator $\mathcal{P}_{0B}$ is a type of projection operator for the state $|\chi_{0B}\rangle$. For simplicity, in this section we dispense with the earlier $\epsilon_B E_{0B}$ notation of representing plus and minus frequencies: E_{0B} in this section can be either plus or minus. We will here summarize the main features of the Bethe–Salpeter perturbation theory, specialized for convenience for our particular example. As mentioned in the Introduction, the concept of "reduced Green's function," introduced earlier in a different context,[10] proves quite useful here as well, because of its organizing effect in the formalism.

For simplicity, we describe the perturbation of a particular nondegenerate level, $|\chi_{0A}\rangle$. "Reduced Green's functions," defined by

$$K_{1A} \equiv \sum_{B \neq A} \mathcal{P}_{0B}/(E_{0A} - E_{0B}) \tag{4.6}$$

and

$$K_{2A} \equiv \sum_{B \neq A} \mathcal{P}_{0B}/(E_{0A} - E_{0B})^2 \tag{4.7}$$

are associated with the given level, $|\chi_{0A}\rangle$. The reduced Green's functions are related to the full Green's function $\mathcal{g}_0(E)$ in a very simple way. It is convenient to express this relationship in terms of the function

$$\mathcal{g}_A(E) \equiv (E - E_{0A})\mathcal{g}_0(E). \tag{4.8}$$

The function $\mathcal{g}_A(E)$ is holomorphic near $E = E_{0A}$, with

$$\mathcal{g}_A(E)|_{E=E_{0A}} = -\mathcal{P}_{0A}, \tag{4.9}$$

$$\left.\frac{d\mathcal{g}_A(E)}{dE}\right|_{E=E_{0A}} = -K_{1A}, \tag{4.10}$$

and

$$\left.\frac{1}{2!}\frac{d^2\mathcal{g}_A(E)}{dE^2}\right|_{E=E_{0A}} = K_{2A}. \tag{4.11}$$

These relations are incorporated in the Laurent expansion

$$\mathcal{g}_0(E) = -\mathcal{P}_{0A}/(E - E_{0A}) - K_{1A} + (E - E_{0A})K_{2A} + \cdots, \tag{4.12}$$

valid near the pole $E = E_{0A}$ of $\mathcal{g}_0(E)$.

Now we consider the more general wave equation $[T(E) + V(E) + \delta V(E)]\chi = 0$, where $\delta V(E)$ is a perturbation. There is an associated Green's function $\mathcal{g}(E)$ obeying

$$[T(E) + V(E) + \delta V(E)]\mathcal{g}(E) = -1, \tag{4.13}$$

and there will be a new frequency eigenfunction expansion

$$\mathcal{g}(E) = -\sum_B \mathcal{P}_B/(E - E_B). \tag{4.14}$$

To facilitate the following derivation of the perturbation equations, we rewrite the Born expansion

$$\mathcal{g}(E) = \mathcal{g}_0(E) + \mathcal{g}_0(E)\delta V(E)\mathcal{g}_0(E) + \cdots$$

or $\mathcal{g}(E)$ in terms of $\mathcal{g}_A(E)$:

$$\mathcal{g}(E) = \frac{\mathcal{g}_A(E)}{E - E_{0A}} + \frac{\mathcal{g}_A(E)\delta V(E)\mathcal{g}_A(E)}{(E - E_{0A})^2} + \frac{\mathcal{g}_A(E)\delta V(E)\mathcal{g}_A(E)\delta V(E)\mathcal{g}_A(E)}{(E - E_{0A})^3} + \cdots. \tag{4.15}$$

Our first perturbation equation is obtained by performing the operation $(1/2\pi i)\oint_C dE \cdots$ on both sides of Eq. (4.15), where C is a contour enclosing E_{0A} and E_A, but no other singular points of either $\mathcal{g}_0(E)$ or $\mathcal{g}(E)$. By the Cauchy integral theorem, we find

$$(1/2\pi i)\oint dE\, \mathcal{g}(E) = \left\{\mathcal{g}_A(E) + \frac{d}{dE}[\mathcal{g}_A(E)\delta V(E)\mathcal{g}_A(E)] + \frac{1}{2!}\frac{d^2}{dE^2}[\mathcal{g}_A(E)\delta V(E)\mathcal{g}_A(E)\delta V(E)\mathcal{g}_A(E)] + \cdots\right\}\Bigg|_{E=E_{0A}}. \tag{4.16}$$

The derivatives in Eq. (4.16) act on functions holomorphic near $E = E_{0A}$. These derivatives are evaluated with the help of Eqs. (4.9)–(4.11). Also, from the eigenfunction expansion (4.14), it can be seen that the contour integral on the left-hand side of Eq. (4.16) has the value $-\mathcal{P}_A$, where $\mathcal{P}_A$ is the exact projection operator for the A th level. Thus

$$-\mathcal{P}_A = -\mathcal{P}_{0A} + K_{1A}\delta V(E_{0A})\mathcal{P}_{0A} + \mathcal{P}_{0A}\delta V(E_{0A})K_{1A} + \mathcal{P}_{0A}\delta V'(E_{0A})\mathcal{P}_{0A} + \cdots,$$

$$\delta V'(E) \equiv \frac{\partial \delta V(E)}{\partial E}. \tag{4.17}$$

If we go back to Eq. (4.15) and perform the operation

$$(1/2\pi i)\oint_C dE\,(E - E_{0A})\cdots$$

on both sides, we encounter similar calculations leading to our second perturbation equation[17]:

$$-\delta E_A \mathscr{P}_A = \mathscr{P}_{0A}\delta V(E_{0A})\mathscr{P}_{0A} - K_{1A}\delta V(E_{0A})\mathscr{P}_{0A}\delta V(E_{0A})\mathscr{P}_{0A} - \mathscr{P}_{0A}\delta V(E_{0A})K_{1A}\delta V(E_{0A})\mathscr{P}_{0A} - \mathscr{P}_{0A}\delta V(E_{0A})\mathscr{P}_{0A}\delta V(E_0)K_{\Delta A} - \mathscr{P}_{0A}\delta V'(E_{0A})\mathscr{P}_{0A}\delta V(E_{0A})\mathscr{P}_{0A} - \mathscr{P}_{0A}\delta V(E_{0A})\mathscr{P}_{0A}\delta V'(E_{0A})\mathscr{P}_{0A} + \cdots. \tag{4.18}$$

The first-order level shift can be obtained quite simply by working with the two perturbation equations (4.17) and (4.18). By Eq.(4.17) $\mathscr{P}_A \approx \mathscr{P}_{0A}$. Equation (4.18) then gives

$$\delta E_A |\chi_{0A}\rangle\langle\overline{\chi_{0A}}| = -|\chi_{0A}\rangle\langle\overline{\chi_{0A}}|\delta V(E_{0A})|\chi_{0A}\rangle\langle\overline{\chi_{0A}}|.$$

Dropping the factor $|\chi_{0A}\rangle$ on the left, and the factor $\overline{\langle\chi_{0A}|}$ on the right gives the desired result

$$\delta E_A = -\overline{\langle\chi_{0A}|}\delta V(E_{0A})|\chi_{0A}\rangle, \tag{4.19}$$

or, more explicitly,

$$\delta E_A = -\int d^3r_2\, d^3r_1\, \overline{\chi_{0A}}(2)\langle \mathbf{r}_2|\delta V(E_{0A})|\mathbf{r}_1\rangle\chi_{0A}(1). \tag{4.20}$$

In spite of its unusual derivation, this result has a familiar form. Note, however, the appearance of the dual wavefunction, $\overline{\chi_{0A}} = \chi^\dagger_{0A}\,\tau_\mu\,\overleftarrow{\Pi}_\mu/2m^2$, where one ordinarily would see the simple adjoint. To compute higher-order corrections, we convert the operator equations (4.17) and (4.18) into c-number statements, for example, by taking their expectation value in the state $|\chi_{0A}\rangle$ and then solve for δE_A by simply dividing the resulting series of c-numbers.

The perturbation theory as described so far is applicable for a general unperturbed system with arbitrary potential $V(E)$. For the special case in which the unperturbed system is the relativistic Kepler problem, with $V(E)$ given by Eq. (4.3), some further simplification is possible. In this case it is advantageous to multiply Eqs. (4.17) and (4.18) on the left and right by $1/r$ before forming an expectation value. This will have the effect of eliminating terms in the resulting c-number equations that have a factor K_{1A} or K_{2A} on the extreme left and/or right. This simplification is a result of the special form of the simultaneous parity and frequency eigenfunctions (3.12)–(3.15): According to these equations, when $\int d^3r\ \overline{\chi_{0B}}\, r^{-1}\chi_{0A}$ is formed, all factors of $r^{1/2}S$ and $r^{1/2}S^{-1}$ disappear, and we obtain integrals that can be evaluated with the help of the simple orthogonality relations (2.15). Reference to the eigenfunction expansions (4.6) and (4.7) of K_{1A} and K_{2A} shows that they contain only states orthogonal to $|\chi_{0A}\rangle$ with respect to the metric r^{-1}. Terms containing a factor K_{1A} or K_{2A} on the extreme left and/or right will be projected out when the expectation value is formed, by virtue of this orthogonality.

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APPENDIX

We introduce the shorthand notation

$$\phi_{En\,J-\frac{1}{2}} \equiv r^{1/2}S\phi_{An\,J-1/2} \equiv r^{1/2}S\phi_{An\,J-(1/2)\,JM}$$

for the (unnormalized) frequency eigenfunctions with $l = J - \frac{1}{2}$. For $n = 2,3,4,\cdots$ there is another level

$$\phi_{E\,n-1\,J+1/2} \equiv r^{1/2}S\phi_{A\,n-1\,J+1/2} \equiv r^{1/2}S\phi_{A\,n-1\,J+(1/2)\,JM}$$

belonging to the same energy. We want to determine linear combinations of the states $\phi_{En\,J-1/2}$ and $\phi_{E\,n-1\,J+1/2}$ that will be parity eigenfunctions. To begin with, we just investigate the dual states $\bar{\phi}_{En\,J-1/2} = (1/2m^2)\,(B\phi_{En\,J-1/2})^\dagger$ and $\bar{\phi}_{E\,n-1\,J+1/2} = (1/2m^2)\,(B\phi_{E\,n-1\,J+1/2})^\dagger$. The effect of $\mathscr{P}$ can then be easily determined using the relation [Eq. (3.10)] $\mathscr{P}\phi = -2m\bar{\phi}^\dagger(-\mathbf{r},t)$.

The calculation of the dual states can be greatly simplified if we assume the appearance of factors $r^{1/2}S^{-1}$ on the right as in Eqs. (3.13) and (3.15). Evidently it is not $B\phi_{EA} = Br^{1/2}S\phi_{AA}$, which is simple, but $r^{-1/2}SBSr^{1/2}\,\phi_{AA}$. Therefore, our calculation of the dual will begin with some operator algebra designed to reduce $r^{-1/2}SBSr^{1/2}$ to a form whose action on the states of interest can be readily determined. Some key equations needed in this connection are

$$B = -(E + a/r) - i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\left(\frac{\partial}{\partial r} + \frac{1-K}{r}\right), \tag{A1}$$

$$S^2 = \cosh\theta + i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\sinh\theta = \frac{|K| + i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}a\epsilon(K)}{(K^2 - a^2)^{1/2}}, \tag{A2}$$

$$\epsilon(K) = +1, \quad K>0, \quad \epsilon(K) = -1, \quad K<0.$$

$$S\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}S = \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}, \tag{A3}$$

and

$$S\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}KS = \frac{|K| + i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}a\epsilon(K)}{(K^2 - a^2)^{1/2}}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}K. \tag{A4}$$

In obtaining these equations repeated use was made of the fact that $\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$ anticommutes with any odd function of K, such as $\epsilon(K)$ for example, and commutes with any even function of K, for example, $|K|$. Note that $\sinh(\frac{1}{2}\theta)$ is an odd function of K and that $\cosh(\frac{1}{2}\theta)$ is an even function of K. The end result of the operator algebra is the identity

$$r^{-1/2}SBSr^{1/2} = -\frac{E|K|}{(K^2 - a^2)^{1/2}} - 2i\eta\left(\frac{\partial}{\partial\rho} + \frac{\frac{3}{2} + \epsilon(K)(K^2 - a^2)^{1/2}}{\rho} - \frac{a\epsilon}{\eta}\frac{\epsilon(K)}{2(K^2 a^2)^{1/2}}\right)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}, \tag{A5}$$

$$\rho \equiv 2\eta r.$$

Proceeding, we introduce the additional abbreviations

$$\phi_{An\,J-1/2} \equiv R_{n\gamma=\nu-1}\,Y_{l=J-1/2} \equiv R_{n\gamma=\nu-1}\,Y_{J-(1/2)\,JM}$$

and

$$\phi_{A\,n-1\,J+1/2} \equiv R_{n-1\,\gamma=\nu}\,Y_{l=j+1/2} \equiv R_{n-1\,\gamma=\nu}\,Y_{J+(1/2)\,JM},$$

where $R_{n\gamma}$ is the function

$$R_{n\gamma} = (2\eta)^{3/2}[(n-1)!/(n+2\gamma)!]^{1/2}\rho^{\gamma-1/2}e^{-\rho/2}L^{2\gamma+1}_{n-1}(\rho) \tag{A6}$$

of Eq. (2.13). In Eq. (A6) $\gamma = \nu - 1$ if $l = J - \frac{1}{2}$ and $\gamma = \nu$ if $l = J + \frac{1}{2}$, where $\nu \equiv (K^2 - \alpha^2)^{1/2}$. Next the operator on the right-hand side of Eq. (A5) is applied to the state $\phi_{AnJ-1/2} = R_{n\gamma=\nu-1} Y_{J-1/2}$. The term $-E|K|(K^2-\alpha^2)^{-1/2}$ in Eq. (A5) just gives a multiple of $\phi_{AnJ-1/2}$ again. However, the $\boldsymbol{\sigma}\cdot\hat{r}$ term in Eq. (A5) has the effect of converting $Y_{J-1/2}$ into $Y_{J+1/2}$ in accordance with the identity

$$\boldsymbol{\sigma}\cdot\hat{r} Y_{lJM}(\hat{r}) = Y_{l'JM}(\hat{r}), \qquad l + l' = 2J. \tag{A7}$$

Note that since K in the $\boldsymbol{\sigma}\cdot\hat{r}$ term acts after $\boldsymbol{\sigma}\cdot\hat{r}$; the value of K is $K = -(J + \frac{1}{2})$ appropriate to the new $l = J + \frac{1}{2}$ angular factor. It is natural to expect that the radial operator in the $\boldsymbol{\sigma}\cdot\hat{r}$ term of Eq. (A5) will adjust the radial factor to match the new angular factor, producing something proportional to $R_{n-1\gamma=\nu} Y_{J+1/2}$. That this indeed happens is a consequence of the identity[18]

$$\rho L_{n-2}^{2\nu+1}(\rho) = -2\nu\left(\frac{\partial}{\partial\rho} + \frac{n-1}{2\nu}\right) L_{n-1}^{2\nu-1}(\rho) \tag{A8}$$

satisfied by the Laguerre polynomials. The end result of acting on $\phi_{AnJ-1/2}$ with the operator is then a simple linear combination of the states $\phi_{AnJ-1/2}$ and $\phi_{An-1J+1/2}$. Taking the adjoint and multiplying on the right with $r^{1/2}S^{-1}$ then gives

$$\bar{\phi}_{EnJ-1/2} = \left\{ -\frac{E|K|}{2m^2\nu}\phi^\dagger_{AnJ-1/2} - \frac{i\eta}{2m^2\nu}[(n-1)(n-1+2\nu)]^{1/2} \times \phi^\dagger_{An-1J+1/2}\right\} r^{1/2}S^{-1}. \tag{A9}$$

The identity

$$\bar{\phi}_{En-1J+1/2} = \left\{ -\frac{E|K|}{2m^2\nu}\phi^\dagger_{An-1J+1/2} + \frac{i\eta}{2m^2\nu}[(n-1)(n-1+2\nu)]^{1/2}\phi^\dagger_{AnJ-1/2}\right\} r^{1/2}S^{-1} \tag{A10}$$

is obtained by a similar calculation. The Laguerre polynomial identity needed this time is[19]

$$\left(\frac{\partial}{\partial\rho} + \frac{1+2\nu}{\rho} - \frac{n-1+2\nu}{2\nu}\right) L_{n-2}^{2\nu+1}(\rho) = \frac{(n-1)(n-1+2\nu)}{2\nu}\frac{L_{n-1}^{2\nu-1}(\rho)}{\rho}. \tag{A11}$$

We can now easily calculate the effect of the parity operator on the frequency eigenfunctions. Relative to the states $\phi_{EnJ-1/2}$ and $\phi_{En-1J+1/2}$ as basis $\mathscr{P}$ has the matrix representative

$$[\mathscr{P}] = (-1)^{J-1/2}\begin{pmatrix} E|K|/m\nu & (i\eta/m\nu)[(n-1)(n-1+2\nu)]^{1/2} \\ (i\eta/m\nu)[(n-1)(n-1+2\nu)]^{1/2} & -E|K|/m\nu \end{pmatrix}. \tag{A12}$$

That the eigenvalues of the matrix multiplying $(-1)^{J-1/2}$ in Eq. (A12) have the expected values ± 1 can be shown with the help of the identity

$$\frac{E^2K^2}{m^2\nu^2} - 1 = \frac{\eta^2}{m^2\nu^2}(n-1)(n-1+2\nu). \tag{A13}$$

This identity is also needed in bringing the eigenvectors to their final forms (3.12)–(3.15). Finally, we note the integrals

$$\int d^3r\, \phi^\dagger_{AB}\phi_{AA} = \delta_{B,A} \tag{A14}$$

[Eq. (2.15)] and

$$\int d^3r\, \phi^\dagger_{AA} E r \phi_{AA} = (n_A + \gamma_A)^2/\alpha \tag{A15}$$

needed to establish the normalization of the eigenfunctions.

[1]L. C. Hostler, J. Math. Phys. **23**, 1179 (1982).

[2]Note that $\bar{\phi}$ defined here differs by a factor $1/2m^2$ from $\bar{\phi}$ of Ref. 1. The new definition is convenient since it will eliminate awkward factors of $2m^2$ elsewhere. Otherwise our notation is the same as in Ref. 1: We use rationalized Gaussian (Heaviside–Lorentz) units, with $\hbar = c = 1$; Lorentz 4-vectors have an imaginary time coordinate, so that our Lorentz metric is the simple $\delta_{\mu\nu}$.

[3]Earlier efforts to unify scalar and spinor electrodynamics seem to have moved in the direction of making the spin zero equations resemble the usual Dirac equation instead of the other way around [N. Kemmer, Proc. Roy. Soc. London A **173**, 91 (1939); M. Neuman and W. H. Furry, Phys. Rev. **76**, 1677 (1949); R. G. Moorhouse, Phys. Rev. **76**, 1691 (1949); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Sec. 9.7 and other references cited therein].

[4]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964); or A. Akhiezer and V. B. Bereztetski, *Quantum Electrodynamics* (Wiley, New York, 1963), 2nd ed.

[5]J. Schwinger, J. Math. Phys. **5**, 1606 (1964).

[6]L. C. Hostler, J. Math.Phys. **11**, 2966 (1970).

[7]L. C. Hostler, J. Math.Phys. **5**, 591 (1964).

[8]T. Kato, Prog. Theor. Phys. **4**, 514 (1949); A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. 2.

[9]G. P. Lepage, Phys. Rev. A **16**, 863 (1977).

[10]Levere C. Hostler, Phys. Rev. **178**, 126 (1969); J. Math. Phys. **16**, 1585 (1975); **12**, 2311 (1971). See also, H. F. Hameka, J. Chem. Phys. **47**, 2728 (1967); **48**, 4810 (1968), erratum; and S. P. Swierkowski and M. Suffczynski, Bull. Acad. Polon. Sci. **XXI**, 285 (1973).

[11]A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, NJ, 1957).

[12]L. C. Biedenharn, Phys. Rev. **126**, 845 (1962).

[13]P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958).

[14]The functions $L_{n-1}^{2\gamma+1}$ are Laguerre polynomials as defined in Herbert Buchholz, *Die Konfluente Hypergeometrische Function* (Springer-Verlag, Berlin, 1953).

[15]This distinction can be quite important. If θ is the eigenvalue, $\sinh(\frac{1}{2}\theta)$ commutes with $\boldsymbol{\sigma}\cdot\hat{r}$; but if θ is the operator, $\sinh(\frac{1}{2}\theta)$ anticommutes with $\boldsymbol{\sigma}\cdot\hat{r}$. Thus we have the identity $T_2\,\phi_{AA}(2)\,\phi^\dagger_{AA}(1)\bar{T}_1 = S_2\,\phi_{AA}(2)\,\phi^\dagger_{AA}(1)\,S_1^{-1}$, in which $S_{2,1}$ denote Schrödinger operators.

[16]We use the definitions of Buchholz (Ref. 14).

[17]Here we require the identity $(1/2\pi i)\oint dE\,(E - E_{0A})\,g(E) = (1/2\pi i)\oint dE\,(E - E_A)g(E) + (E_A - E_{0A})(1/2\pi i)\oint dE\,g(E) = 0 + \delta E_A(-\mathscr{P}_A)$, since $(E - E_A)g(E)$ is holomorphic near $E = E_A$.

[18]Equation (A8) is readily verified by expressing each term using the integral representation

$$L_n^\mu(\rho) = (1/2\pi i)\oint dt\, e^{-t\rho}(1+t)^{n+\mu}t^{-(n+1)}$$

[Ref. 14, p. 135, Eq. (2)]. The term $\rho L_{n-2}^{2\nu+1}(\rho)$ is treated by integration by parts:

$$\rho L_{n-2}^{2\nu+1} = -(1/2\pi i)\oint [d(e^{-t\rho})/dt](1+t)^{n-1+2\nu}t^{-(n-1)}$$
$$= (1/2\pi i)\oint dt e^{-t\rho} d[(1+t)^{n-1+2\nu}t^{-(n-1)}]/dt.$$

Verification of the relation (A8) between the Laguerre polynomials is thereby reduced to verifying the elementary identity

$$d[(1+t)^{n-1+2\nu}t^{-(n-1)}]/dt = [2\nu t - (n-1)](1+t)^{n-2+2\nu}t^{-n}.$$

[19]Equation (A11) can be verified using the method of Ref. 18.

Infinitesimal symmetries and conserved currents for nonlinear Dirac equations

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The complete algebras of infinitesimal symmetries for Dirac equations, Dirac equations with nonvanishing rest mass, and Dirac equations with a nonlinear term have been established. In addition, conserved currents associated to new symmetries have been constructed.

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1. INTRODUCTION

By the systematic approach of Harrison and Estabrook[1,2] and the use of symbolic computations,[3,4] we are able to construct the complete algebra of infinitesimal symmetries for four different types of Dirac equations, i.e., Dirac equations with vanishing rest mass, Dirac equations with nonvanishing rest mass, Dirac equations with a nonlinear term, and nonlinear Dirac equations without rest mass. The results obtained here contain the previous results by Anderson and Ibragimov[5] and complete the results of Steeb *et al.*[6,7] for nonlinear Dirac equations. Due to this systematic approach, we are able to prove a conjecture made by Steeb *et al.*[6] In addition, conserved currents associated to new symmetries are constructed.

The infinitesimal symmetries of a closed ideal I in n-dimensional space, generated by a set of differential forms

$$\alpha(1),\ldots,\alpha(m)$$

are vector fields

$$V = V^i \frac{\partial}{\partial x^i} \quad \text{(summation convention)} \tag{1.1}$$

such that

$$\mathcal{L}_V I \subset I, \tag{1.2}$$

where $\mathcal{L}_V$ denotes the Lie derivative by the vector field V.[1,2] From (1.2), we see that V has to satisfy

$$\mathcal{L}_V \alpha(i) + \gamma(i,j) \wedge \alpha(j) = 0 \quad (i := 1{:}m), \tag{1.3}$$

where $\gamma(i,j)$ are suitable differential forms and $\gamma(i,j) \wedge \alpha(j)$ denotes the wedge product of $\gamma(i,j)$ and $\alpha(j)$ summed over j.

The first purpose of this paper is the construction of the set of generators of the local symmetries for Dirac equations,

$$\sum_{k=1}^{3} \hbar \frac{\partial}{\partial x_k}(\gamma_k \psi) - i\hbar \frac{\partial}{\partial x_4}(\gamma_4 \psi) + m_0 c \psi + n_0 \psi(\bar{\psi}\psi) = 0, \tag{1.4}$$

where in (1.4) $x_4 = ct$, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ (T means transpose), $\bar{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are 4×4 matrices defined by

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1.4a}$$

Introducing the quantity

$$\lambda = \hbar / m_0 c,$$

we obtain

$$\lambda \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\gamma_k \psi) - \lambda i \frac{\partial}{\partial x_4}(\gamma_4 \psi) + \psi + \lambda^3 \epsilon \psi(\bar{\psi}\psi) = 0. \tag{1.5}$$

We shall construct the set of generators of the local symmetries of (1.5) for four different problems.

(1) $\epsilon = 0, \lambda^{-1} = 0$, i.e., Dirac equations with vanishing rest mass;

(2) $\epsilon = 0, \lambda^{-1} \neq 0$, i.e., Dirac equations with nonvanishing rest mass; (1.6)

(3) $\epsilon \neq 0, \lambda^{-1} \neq 0$, i.e., Dirac equations with a nonlinear term;

(4) $\epsilon \neq 0, \lambda^{-1} = 0$, i.e., Dirac equations with a nonlinear term and vanishing rest mass.

If we now put $\psi_j = u^j + iv^j$ ($j = 1,\ldots,4$) into (1.5), we arrive at a system of eight coupled partial differential equations:

$$\begin{aligned}
&\lambda v_1^4 - \lambda u_2^4 + \lambda v_3^3 + \lambda v_4^1 + (1 + \lambda^3 \epsilon K) u^1 = 0, \\
&\lambda v_1^3 + \lambda u_2^3 - \lambda v_3^4 + \lambda v_4^2 + (1 + \lambda^3 \epsilon K) u^2 = 0, \\
&-\lambda v_1^2 + \lambda u_2^2 - \lambda v_3^1 - \lambda v_4^3 + (1 + \lambda^3 \epsilon K) u^3 = 0, \\
&-\lambda v_1^1 - \lambda u_2^1 + \lambda v_3^2 - \lambda v_4^4 + (1 + \lambda^3 \epsilon K) u^4 = 0, \\
&-\lambda u_1^4 - \lambda v_2^4 - \lambda u_3^3 - \lambda u_4^1 + (1 + \lambda^3 \epsilon K) v^1 = 0, \\
&-\lambda u_1^3 + \lambda v_2^3 + \lambda u_3^4 - \lambda u_4^2 + (1 + \lambda^3 \epsilon K) v^2 = 0, \\
&\lambda u_1^2 + \lambda v_2^2 + \lambda u_3^1 + \lambda u_4^3 + (1 + \lambda^3 \epsilon K) v^3 = 0, \\
&\lambda u_1^1 - \lambda v_2^1 - \lambda u_3^2 + \lambda u_4^4 + (1 + \lambda^3 \epsilon K) v^4 = 0,
\end{aligned} \tag{1.7}$$

where in (1.7), $u_j^i = \partial u^i / \partial_{x^j}$, $v_j^i = \partial v^i / \partial_{x^j}$,

$$K = (u^1)^2 + (u^2)^2 + (v^1)^2 + (v^2)^2 - (u^3)^2 - (u^4)^2 - (v^3)^2 - (v^4)^2.$$

We can construct an ideal I of differential 1-forms asso-

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ciated to (1.7) by using the contact 1-forms[8,9]

$$\alpha(i) = du^i - u_1^i\,dx^1 - u_2^i\,dx^2 - u_3^i\,dx^3 - u_4^i\,dx^4 \quad (i := 1{:}4),$$
$$\alpha(4+i) = dv^i - v_1^i\,dx^1 - v_2^i\,dx^2 - v_2^i\,dx^3 - v_4^i\,dx^4 \quad (i := 1{:}4),$$
$$d\alpha(1),\ldots,d\alpha(8), \tag{1.8}$$

and substitution of u_4^i, v_4^i $(i := 1{:}4)$ from (1.7) into (1.8). So we consider an ideal I in 36-dimensional space $(= \mathbb{R}^{36})$,

$$\mathbb{R}^{36} = \{(x_1,\ldots,x_4, u^1,\ldots,u^4, v^1,\ldots,v^4, u_1^1,\ldots,u_3^1,\ldots,v_1^4,\ldots v_3^4)\}, \tag{1.9}$$

generated by (1.8), (1.8a).

We derive the infinitesimal symmetries of (1.7), (1.8), doing all computation with the use of the symbolic Language REDUCE[3] on a DEC computer system.

We used a set of beautiful subroutines constructed by Gragert[4] which enabled us to do differential geometric calculations on the computer. In Sec. 2, we shall state the results for the complete algebras of infinitesimal symmetries for the four different problems (1.5), (1.6). The detailed computations can be found in Ref. 10. The second purpose of this paper is the construction of conserved currents, associated to the results of Sec. 2. This has been done in Sec. 3, using the local jet bundle formalism.[8,9]

2. THE COMPLETE ALGEBRAS OF INFINITESIMAL SYMMETRIES FOR DIRAC EQUATIONS

In this section, we state the results of the computation of the complete algebras of infinitesimal symmetries of Dirac equations (1.5) and (1.6).

Theorem 1: $(\epsilon = 0;\ \lambda^{-1} = 0)$ The complete algebra of infinitesimal symmetries for Dirac equations with vanishing rest mass is spanned by 23 generators. Moreover, there is a continuous part, generated by functions $V^{u_1},\ldots,V^{v_4}$ dependent on $x^1,\ldots,x^4$ satisfying Dirac equations, due to the linearity of the problem (1.5), (1.6a).[1] This algebra contains the 15 generators of the conformal group and 8 generators

$$\begin{aligned}
X^1 &= u^4\partial_{u^1} - u^3\partial_{u^2} - u^2\partial_{u^3} + u^1\partial_{u^4} - v^4\partial_{v^1} + v^3\partial_{v^2} + v^2\partial_{v^3} - v^1\partial_{v^4},\\
X^2 &= v^1\partial_{u^1} + v^2\partial_{u^2} + v^3\partial_{u^2} + v^4\partial_{u^4} - u^1\partial_{v^1} - u^2\partial_{v^2} - u^3\partial_{v^3} - u^4\partial_{v^4},\\
X^3 &= v^4\partial_{u^1} - v^3\partial_{u^2} - v^2\partial_{u^3} + v^1\partial_{u^4} + u^4\partial_{v^1} - u^3\partial_{v^2} - u^2\partial_{v^3} + u^1\partial_{v^4},\\
X^4 &= u^1\partial_{u^1} + u^2\partial_{u^2} + u^3\partial_{u^3} + u^4\partial_{u^4} + v^1\partial_{v^1} + v^2\partial_{v^2} + v^3\partial_{v^3} + v^4\partial_{v^4},\\
X^5 &= u^2\partial_{u^1} - u^1\partial_{u^2} - u^4\partial_{u^3} + u^3\partial_{u^4} - v^2\partial_{v^1} + v^1\partial_{v^2} + v^4\partial_{v^3} - v^3\partial_{v^4},\\
X^6 &= u^3\partial_{u^1} + u^4\partial_{u^2} + u^1\partial_{u^3} + u^2\partial_{u^4} + v^3\partial_{v^1} + v^4\partial_{v^2} + v^1\partial_{v^3} + v^2\partial_{v^4},\\
X^7 &= v^2\partial_{u^1} - v^1\partial_{u^2} - v^4\partial_{u^3} + v^3\partial_{u^4} + u^2\partial_{v^1} - u^1\partial_{v^2} - u^4\partial_{v^3} + u^3\partial_{v^4},\\
X^8 &= v^3\partial_{u^1} + v^4\partial_{u^2} + v^1\partial_{u^3} + v^2\partial_{u^4} - u^3\partial_{v^1} - u^4\partial_{v^2} - u^1\partial_{v^3} - u^2\partial_{v^4}.
\end{aligned} \tag{2.1}$$

Theorem 2: $(\epsilon = 0,\ \lambda^{-1} \neq 0)$ The complete algebra of infinitesimal symmetries for Dirac equations with nonvanishing rest mass is spanned by 14 generators. Moreover, there is a continuous part generated by functions $V^{u_1},\ldots,V^{v_4}$ dependent on $x^1,\ldots,x^4$ satisfying Dirac equations with nonvanishing rest mass (1.5), (1.6b) due to the linearity of the problem. This algebra contains the 10 generators of the Poincaré group and the generators X^1, X^2, X^3, X^4, of Eq. (2.1).

The results obtained in Theorem 1 and Theorem 2 are in agreement with those of Anderson and Ibragimov.[5]

Theorem 3: $(\epsilon \neq 0,\ \lambda^{-1} \neq 0)$ The complete algebra of infinitesimal symmetries for nonlinear Dirac equations with nonvanishing rest mass is spanned by 13 generators. The generators, in this case, are the 10 generators of the Poincaré group and X^1, X^2, X^3, of Eq. (2.1).

The result obtained by Steeb *et al.*,[6] i.e., X^2 is a generator, in this case, is contained in ours. Moreover, in deriving our results,[10] we proved the conjecture that the infinitesimal symmetries are at most linear in $u^1,\ldots,v^4$.[6]

Theorem 4: $(\epsilon \neq 0,\ \lambda^{-1} = 0)$ The complete algebra of infinitesimal symmetries for nonlinear Dirac equations with vanishing rest mass is spanned by 14 generators. The generators are 10 generators of the Poincaré group, X^1, X^2, X^3 as in (2.1) and U^1, i.e.,

$$U^1 = x^1\partial_{x^1} + x^2\partial_{x^2} + x^3\partial_{x^3} + x^4\partial_{x^4} - \frac{u^1}{2}\partial_{u^1} - \frac{u^2}{2}\partial_{u^2} - \frac{u^3}{2}\partial_{u^3} - \frac{u^4}{2}\partial_{u^4} - \frac{v^1}{2}\partial_{v^1} - \frac{v^2}{2}\partial_{v^2} - \frac{v^3}{2}\partial_{v^3} - \frac{v^4}{2}\partial_{v^4}.$$

U^1 is the generator of a scaling transformation.

3. CONSERVED CURRENTS FOR NONLINEAR DIRAC EQUATIONS

In this section, conserved currents associated to the infinitesimal symmetries X^1, X^2, X^3 of Theorem 3 of Sec. 2 shall be constructed. Throughout this section, we shall use the local jet bundle formalism.[8,9]

The generators X^1, X^2, X^3 are vertical vector fields on $T(E) = \mathbb{R}^{12}$, generated by $\partial_{x^1},\ldots,\partial_{x^4}, \partial_{u^1},\ldots,\partial_{v^4}$; i.e.,

$$\begin{aligned}
&\pi_M X^j = 0 \quad (j = 1, 2, 3),\\
X^1 &= u^4\partial_{u^1} - u^3\partial_{u^2} - u^2\partial_{u^3} + u^1\partial_{u^4} - v^4\partial_{v^1} + v^3\partial_{v^2} + v^2\partial_{v^3} - v^1\partial_{v^4},\\
X^2 &= v^1\partial_{u^1} + v^2\partial_{u^2} + v^3\partial_{u^2} + v^4\partial_{u^4} - u^1\partial_{v^1} - u^2\partial_{v^2} - u^3\partial_{v^3} - u^4\partial_{v^4},\\
X^3 &= v^4\partial_{u^1} - v^3\partial_{u^2} - v^2\partial_{u^3} + v^1\partial_{u^4} + u^4\partial_{v^1} - u^3\partial_{v^2} - u^2\partial_{v^3} + u^1\partial_{v^4}.
\end{aligned} \tag{3.1}$$

In fact, we need the prolonged fields p^1X^1, p^1X^2, p^1X^3 on $J^1(E)$, which can be obtained from (3.1).[8,9]

Let $L(u, v, u_i, v_i)$ be the Lagrangian, defined on $J^1(E)$ by

$$
\begin{aligned}
L = (&-u^4v_1^1 + v^4u_1^1 - u^3v_1^2 + v^3u_1^2 - u^2v_1^3 + v^2u_1^3 - u^1v_1^4 + v^1u_1^4 \\
&- v^4v_2^1 - u^4u_2^1 + v^3v_2^2 + u^3u_2^2 - v^2v_2^3 - u^2u_2^3 + v^1v_2^4 + u^1u_2^4 \\
&- u^3v_3^1 + v^3u_3^1 + u^4v_3^2 - v^4u_3^2 - u^1v_3^3 + u^2v_3^4 - v^2u_3^4 \\
&- u^1v_4^1 + v^1u_4^1 - u^2v_4^2 + v^2u_4^2 - u^3v_4^3 + v^1u_3^3 + v^3u_4^3 - u^4v_4^4 + v^4u_4^4) \\
&- K(1 + \tfrac{1}{2}\lambda^3\epsilon K),
\end{aligned} \tag{3.2}
$$

where $(x, u, v, u_i, v_i) = (x^1,...,x^4,u^1,...,v^4,u_1^1,...,v_4^4)$ are local coordinates on $J^1(E) = \mathbb{R}^{44}$.

An easy calculation shows that the Euler–Lagrange equations associated with (3.2), i.e.,

$$
\frac{\partial}{\partial x^a}\left(\frac{\partial L}{\partial z_a^A}\right) - \frac{\partial L}{\partial z^A} = 0, \tag{3.3}
$$

are just nonlinear Dirac equations (1.7). In this section, we use the summation convention ($A = 1,...,8$; $a = 1,...,4$) if an index occurs twice.

We introduce the Cartan form θ defined by[8]

$$
\theta = L\omega + (\partial_A^a L)\theta^A \wedge \omega_a, \tag{3.4}
$$

where in (3.4)

$$
\begin{aligned}
&\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \\
&\omega_a = \partial_a \lrcorner \omega, \\
&\partial_a = \partial_{x^a}, \partial_A = \partial_{z^A}, \partial_A^a = \partial_{z_a^A}, \\
&\theta^A = dz^A - z_a^A\, dx^a
\end{aligned} \tag{3.4a}
$$

(z_a^A referring to either u_a^i or v_a^i).

A somewhat lengthy calculation[10] leads to

Lemma 1:

$$
\mathscr{L}_{p^1X^i}(\theta) = 0 \quad (i = 1, 2, 3). \tag{3.5}
$$

Due to the relation

$$
\mathscr{L}_{p^1X^i}\theta = p^1X^i \lrcorner d\theta + d(p^1X^i \lrcorner \theta) = 0 \quad (i = 1.\ 2,\ 3)
$$

and since[9,10]

$$
p^1X^i \lrcorner d\theta\,|_I = 0 \quad (i = 1, 2, 3),
$$

we derive

$$
d(p^1X^i \lrcorner \theta) \in I \quad (i = 1, 2, 3), \tag{3.6}
$$

i.e., $p^1X^i \lrcorner \theta$ is a conserved current to (1.8). We now formulate the following:

Theorem 5: The conserved currents associated to the infinitesimal symmetries X^1, X^2, X^3 for problem (1.5), (1.6b) are given by

$$
\begin{aligned}
X^1\colon\ &+ 2(u^4v^4 - u^3v^3 - u^2v^2 + u^1v^1)\, dx^2\, dx^3\, dx^4 \\
&- ((u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 - (v^1)^2 \\
&- (v^2)^2 + (v^3)^2 + (v^4)^2)\, dx^1\, dx^3\, dx^4 \\
&+ 2(u^4v^3 + u^3v^4 - u^2v^1 - u^1v^2)\, dx^1\, dx^2\, dx^4 \\
&- 2(u^4v^1 - u^3v^2 - u^2v^3 + u^1v^4)\, dx^1\, dx^2\, dx^3,
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
X^2\colon\ &+ 2(v^1v^4 + v^2v^3 + u^1u^4 + u^2u^3)\, dx^2\, dx^3\, dx^4 \\
&- 2(-u^4v^1 + u^3v^2 - u^2v^3 + u^1v^4)\, dx^1\, dx^3\, dx^4 \\
&+ 2(v^1v^3 - v^2v^4 + u^1u^3 - u^2u^4)\, dx^1\, dx^2\, dx^4 \\
&- ((u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 \\
&+ (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2)\, dx^1\, dx^2\, dx^3, \\
X^3\colon\ &(-(u^1)^2 + (u^2)^2 + (u^3)^2 - (u^4)^2 \\
&+ (v^1)^2 - (v^2)^2 - (v^3)^2 + (v^4)^2)\, dx^2\, dx^3\, dx^4 \\
&- 2(-u^4v^4 - u^3v^3 + u^2v^2 + u^1v^1)\, dx^1\, dx^3\, dx^4 \\
&+ 2(v^3v^4 - v^2v^1 - u^3u^4 + u^1u^2)\, dx^1\, dx^2\, dx^4 \\
&- 2(v^1v^4 - v^3v^2 - u^1u^4 + u^2u^3)\, dx^1\, dx^2\, dx^3.
\end{aligned}
$$

4. CONCLUSION

Using symbolic computations, we have been able to establish the complete algebras of infinitesimal symmetries for nonlinear Dirac equations. Moreover, conserved currents, associated to new symmetries, have been constructed.

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[1] B. K. Harrison and F. B. Estabrook, "Geometric approach to invariance groups and solutions of partial differential systems," J. Math. Phys. **12**, 653–666 (1971).

[2] *Geometric Appoaches to Differential Equations, Lecture Notes in Mathematics*, Vol. 810, edited by R. Martini (Springer, New York, 1980).

[3] A. C. Hearn, *Reduce 2 User's Manual* (University of Utah, Salt Lake City, 1973).

[4] P. K. H. Gragert, "Symbolic Computations in Prolongation Theory," Ph.D. thesis, Twente University of Technology, The Netherlands, 1981.

[5] R. L. Anderson and N. H. Ibragimov, *Lie-Bäcklund Transformations in Applications* (SIAM Studies in Applied Mathematics, Philadelphia, 1979).

[6] W.-H. Steeb, W. Erig, and W. Strampp, "Symmetries and the Dirac equation," J. Math. Phys. **22**, 970–973 (1981).

[7] W.-H. Steeb, W. Erig, and W. Strampp, "Similarity solutions of nonlinear Dirac equations and conserved currents," J. Math. Phys. **23**, 145–153 (1982).

[8] W. F. Shadwick, "The Bäcklund problem, Symmetries and Conservation Laws for Some Nonlinear Differential Equations," Ph.D. thesis, King's College, London, 1979.

[9] F. A. E. Pirani, D. C. Robinson, and W. F. Shadwick, "Local Jet Bundle Formulation of Bäcklund Transformations," in *Mathematical Physics Studies 1* (Reidel, Boston, 1979).

[10] P. H. M. Kersten, "Infinitesimal Symmetries and Conserved Currents for Nonlinear Dirac Equations," Memorandum 398, Twente University of Technology, The Netherlands, 1982.